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A Text Book For Engineering Mathematics, Vol. - 2

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Preface

Engineering mathematics serves as the foundation for a broad range of disciplines within science and engineering. A solid understanding of its principles is essential for students pursuing fields such as engineering, physics, computer science, and applied mathematics. This book is designed to bridge the gap between abstract mathematical theory and practical application, providing students with the tools and insight required to solve real-world problems with confidence and precision. The purpose of this volume is to offer a comprehensive and accessible resource for both learning and teaching core concepts in linear algebra and vector spaces—areas that are crucial to many engineering applications. It is intended for students across various domains who require a rigorous yet approachable treatment of these mathematical topics.
The content of this book is organized into the following seven chapters:

1. **Matrices (I)**
2. **Determinants**
3. **Matrices (II)**
4. **Systems of Linear Equations**
5. **Eigenvalues and Eigenvectors**
6. **Vector Spaces**
7. **Inner Product Spaces and Orthogonality**

Each chapter has been developed to be as self-contained as possible, allowing instructors the flexibility to tailor the order and depth of topics to match their specific course requirements. Prerequisites are clearly indicated at the start of each chapter, helping students to navigate the material in a structured and logical manner. To enhance the teaching and learning experience, this book incorporates several key features:

- **Clarity through Simplicity**: Examples are carefully chosen and kept straightforward to ensure that students first master the fundamental ideas before moving on to more advanced applications.
- **Modular Structure**: Each chapter and section is designed to stand alone, giving educators the freedom to adapt the material to various teaching styles and academic programs.
- **Self-Contained Presentation**: Most topics are fully developed within the text. In the few instances where deeper theoretical treatment is beyond the book's scope, appropriate references are provided.
- **Progressive Complexity**: Topics are introduced in a gradual, step-by-step fashion—from basic principles to more complex ideas—to help students build both understanding and confidence.
- **Standardized Notation**: Contemporary and widely accepted notation is used throughout the book to ensure consistency and facilitate cross-referencing with other resources.

This volume is part of an ongoing effort to make engineering mathematics more accessible, relevant, and engaging. Feedback from both students and instructors is welcome and appreciated, and will be instrumental in guiding improvements in future editions.

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Acknowledgement

I would like to express my heartfelt gratitude to **Swami Vivekananda University, Kolkata, India**, for their invaluable support and encouragement in the creation of this book, *A Textbook for Engineering Mathematics, Vol – 2*. The university's unwavering commitment to promoting academic excellence and fostering a research-driven environment has been instrumental in the development of this work.

The collaborative academic atmosphere, along with the extensive resources and infrastructure provided by the university, has enabled us to explore and present key concepts and recent developments in engineering mathematics with depth and clarity. This supportive setting has greatly contributed to the quality and relevance of the material presented in this volume, ensuring it meets the evolving needs of engineering students.

I am particularly grateful for the university's vision and dedication to innovation in teaching and learning, which have inspired and guided the preparation of this textbook. Their encouragement has played a vital role in making this book a comprehensive and accessible resource for both students and educators.

It is my sincere hope that this book, a reflection of our shared dedication to knowledge and progress, will serve as a valuable academic tool for the students of Swami Vivekananda University and the wider academic community. I look forward to continuing this partnership and contributing further to the advancement of education and research in engineering mathematics.

Thank you to everyone at Swami Vivekananda University who has supported this endeavor. Your contributions are deeply appreciated.

Sincerely,

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Contents

Chapter 1

Matrices (I)

1.1 Introduction .

Matrix is provides us with a very powerful mathematical tool which has wide application in science and technology. In this chapter we have scope to study the matrix over real number only. Many theorems are illustrated by a good number of examples so that engineers may feel comfort in using matrix towards any of their work. System of linear equations is an interesting part of this chapter. Here we deal with the system where number of unknown quantities and number of equations may not be same which may appear in several practical problems. We introduce a very scientific method to find whether a system is solvable at all and, if solvable, how to solve the system.

1.2 Matrices Definition

Matrices are the ordered rectangular array of numbers, which are used to express linear equations. A matrix has rows and columns. we can also perform the mathematical operations on matrices such as addition, subtraction, multiplication of matrix. Suppose the number of rows is m and columns is n, then the matrix is represented as m × n matrix.

$$\begin{bmatrix} a_1 & \cdots & a_n \\ \vdots & \ddots & \vdots \\ a_m & \cdots & a_{mn} \end{bmatrix}$$

1.3 Types of Matrices

There are different types of matrices. Let's see some of the examples of different types of matrices

- **Square Matrix :** A matrix in which the number of rows is equal to the number of columns.
  Example : $\begin{bmatrix} 2 & 1 & 4 \\ 5 & 6 & 7 \\ 32 & 1 & 2 \end{bmatrix}$ is a square matrix of order 3 x 3.
- **Row Matrix and Column Matrix :** A matrix having only one row is called a row matrix.
  Example : $[2 \quad 0 \quad 6]$ is a 1 x 3 row matrix.
  A matrix having only one column is called a column matrix.
  Example : $\begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix}$ is a 3 x 1 column matrix.

- **Zero matrix / Null matrix :** A matrix of any order whose all elements are zero is called a null matrix and is denoted by O.

Example : $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

- **Diagonal matrix :** A square matrix whose elements except those in the leading diagonal are zero is called a diagonal matrix. i.e, $a_{ij} = 0$ for all $i \neq j$.

Example : $\begin{bmatrix} 6 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

- **Identity matrix / Unit Matrix :** A square matrix $A = [a_{ij}]_{n\times n}$ is called an identity matrix or unit matrix if

(i) $a_{ij} = 0 \; for\ all\ i \neq j\ and\ (ii)\ a_{ii} = 1 \; for\ all\ i.$

Example : $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- **Upper triangular matrix :** A square matrix $A = [a_{ij}]_{n\times n}$ is said to be an upper triangular matrix if $a_{ij} = 0 \; for\ all\ i > j$; i.e , all elements below the main diagonal are zero.

Example : $\begin{bmatrix} 6 & -1 & 5 \\ 0 & 9 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

- **Lower triangular matrix :** A square matrix $A = [a_{ij}]_{n\times n}$ is said to be an lower triangular matrix if $a_{ij} = 0 \; for\ all\ i < j$; i.e , all elements above the main diagonal are zero.

Example : $\begin{bmatrix} 6 & 0 & 0 \\ 2 & 9 & 0 \\ 8 & -1 & 2 \end{bmatrix}$

1.4 Algebraic Operations On Matrices :

- **Equality of two Matrices :** Two matrices A and B are said to be equal if and only if

(i) A and B have the same order and

(ii) Each element of A is equal to the corresponding element of B.

Example : Two Matrices $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\ and\ B = \begin{bmatrix} p & q & r \\ s & t & u \end{bmatrix}$ are equal if

$a = p, b = q, c = r, d = s, e = t,\ f = u$

- **Addition of Matrices :** Two matrices $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ are said to be conformable for addition if they are of the same order. The sum of the two matrices A and B is then defined as the matrix each of whose elements is the sum of the corresponding elements of A and B.

A + B = $[a_{ij}]_{m\times n} + [b_{ij}]_{m\times n} = [c_{ij}]_{m\times n}$ where $c_{ij} = a_{ij} + b_{ij}$ for all i and j.

Example 1: If A = $\begin{bmatrix} 2 & 0 & 9 \\ -4 & 1 & 6 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & 7 & 0 \\ 3 & -4 & -5 \end{bmatrix}$ then

$$A + B = \begin{bmatrix} 2+1 & 0+7 & 9+0 \\ -4+3 & 1-4 & 6-5 \end{bmatrix} = \begin{bmatrix} 3 & 7 & 9 \\ -1 & -3 & 1 \end{bmatrix}$$

1.5 Properties of Matrix Addition :

(i) Commutativity : For any two matrices A and B of the same order , A + B = B + A

(ii) Associativity : If A, B and C are three matrices of the same order, then

$$(A + B) + C = A + (B + C)$$

(iii) Existence of additive identity : The Null Matrix is the identity element for matrix addition.I.e,

$$A + O = O + A$$

(iv) Existence of additive inverse : For every matrix $A = [a_{ij}]_{m\times n}$ there exist a matrix $-A = [-a_{ij}]_{m\times n}$ such that $A + (-A) = O = (-A) + A$

Example 2 : Find x , y, z and t if $2\begin{bmatrix} x & z \\ y & t \end{bmatrix} + 3\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = 3\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$.

Solution : $2\begin{bmatrix} x & z \\ y & t \end{bmatrix} + 3\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = 3\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$

or, $2\begin{bmatrix} x & z \\ y & t \end{bmatrix} + \begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 15 \\ 12 & 18 \end{bmatrix}$

or, $2\begin{bmatrix} x & z \\ y & t \end{bmatrix} = \begin{bmatrix} 9 & 15 \\ 12 & 18 \end{bmatrix} - \begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix}$

or, $2\begin{bmatrix} x & z \\ y & t \end{bmatrix} = \begin{bmatrix} 9-3 & 15-(-3) \\ 12-0 & 18-0 \end{bmatrix}$

or, $2\begin{bmatrix} x & z \\ y & t \end{bmatrix} = \begin{bmatrix} 6 & 18 \\ 12 & 18 \end{bmatrix}$

Or, $\begin{bmatrix} x & z \\ y & t \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 6 & 9 \end{bmatrix}$

Ans : $x = 3, y = 6, z = 9, t = 9$

Example 3 : Determine the matrices A and B where

$$A + 2B = \begin{bmatrix} 1 & 2 & 0 \\ 6 & -3 & 3 \\ -5 & 3 & 1 \end{bmatrix} \text{ and } 2A - B = \begin{bmatrix} 2 & -1 & 5 \\ 2 & -1 & 6 \\ 0 & 1 & 2 \end{bmatrix}$$

Solution : Given ,

$$A + 2B = \begin{bmatrix} 1 & 2 & 0 \\ 6 & -3 & 3 \\ -5 & 3 & 1 \end{bmatrix} \text{.......... (i)}$$

$$2A - B = \begin{bmatrix} 2 & -1 & 5 \\ 2 & -1 & 6 \\ 0 & 1 & 2 \end{bmatrix} \text{.............. (ii)}$$

Multiplying equation (i) by 2 we get,

$$2A + 4B = \begin{bmatrix} 2 & 4 & 0 \\ 12 & -6 & 6 \\ -10 & 6 & 2 \end{bmatrix} \text{..............(iii)}$$

Substracting (ii) from (iii)

$$2A + 4B - (2A - B) = \begin{bmatrix} 2 & 4 & 0 \\ 12 & -6 & 6 \\ -10 & 6 & 2 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 5 \\ 2 & -1 & 6 \\ 0 & 1 & 2 \end{bmatrix}$$

$$5B = \begin{bmatrix} 2-2 & 4-(-1) & 0-5 \\ 12-2 & -6-(-1) & 6-6 \\ -10-0 & 6-1 & 2-2 \end{bmatrix}$$

$$5B = \begin{bmatrix} 0 & 5 & -5 \\ 10 & -5 & 0 \\ -10 & 5 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -1 & 0 \\ -2 & 1 & 0 \end{bmatrix}$$

By putting the value of B in (i)

$$A + 2\begin{bmatrix} 0 & 1 & -1 \\ 2 & -1 & 0 \\ -2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 6 & -3 & 3 \\ -5 & 3 & 1 \end{bmatrix}$$

$$A + \begin{bmatrix} 0 & 2 & -2 \\ 4 & -2 & 0 \\ -4 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 6 & -3 & 3 \\ -5 & 3 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 6 & -3 & 3 \\ -5 & 3 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 2 & -2 \\ 4 & -2 & 0 \\ -4 & 2 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ -1 & 1 & 1 \end{bmatrix}$$

1.6 Multiplication of Matrices by a scalar:

If $A = [a_{ij}]$ be an $m \times n$ matrix and k be any number called a scalar. Then the matrix obtained by multiplying every element of A by k is called the scalar multiple of A by k and is denoted by kA.

Thus, $kA = [ka_{ij}]_{m \times n}$

For Example: if $A = \begin{bmatrix} 1 & 2 & 5 \\ -2 & 3 & 4 \\ 1 & 2 & -1 \end{bmatrix}$, $then\ 3A = \begin{bmatrix} 3 & 6 & 15 \\ -6 & 9 & 12 \\ 3 & 6 & -3 \end{bmatrix}$

Properties of scalar multiplication:

Various properties of scalar multiplication are stated and proved in the following theorem.

Theorem: If $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$ are two matrices and k, l are scalars, then

(i) $k(A + B) = kA + kB$

(ii) $(k + l)A = kA + lA$

(iii) $(kl)A = k(lA) = l(kA)$

(iv) $(-k)A = -(kA) = k(-A)$

(v) $1A = A$

(vi) $(-1)A = -A$

1.7 Subtraction of Matrices:

Definition: For two matrices A and B of the same order, the subtraction of matrix B from matrix A is denoted by $A - B$ and is defined as $A - B = A + (-B)$.

For Example :

If $A = \begin{bmatrix} -3 & 2 & 1 \\ 1 & -4 & 7 \end{bmatrix}$ $and\ B = \begin{bmatrix} 3 & 5 & -2 \\ -1 & 4 & -2 \end{bmatrix}$ $, then$

$A - B = \begin{bmatrix} -3 & 2 & 1 \\ 1 & -4 & 7 \end{bmatrix} - \begin{bmatrix} 3 & 5 & -2 \\ -1 & 4 & -2 \end{bmatrix} = \begin{bmatrix} -3-3 & 2-5 & 1+2 \\ 1+1 & -4-4 & 7+2 \end{bmatrix} = \begin{bmatrix} -6 & -3 & 3 \\ 2 & -8 & 9 \end{bmatrix}$.

***Example* 4: Find a matrix A such that $2A - 3B + 5C = O$, *where* $B = \begin{bmatrix} -2 & 2 & 0 \\ 3 & 1 & 4 \end{bmatrix}$ *and* $C = \begin{bmatrix} 2 & 0 & -2 \\ 7 & 1 & 6 \end{bmatrix}$.**

Solution : We have,

$2A - 3B + 5C = O$

$2A = 3B - 5C$

$2A = 3\begin{bmatrix} -2 & 2 & 0 \\ 3 & 1 & 4 \end{bmatrix} - 5\begin{bmatrix} 2 & 0 & -2 \\ 7 & 1 & 6 \end{bmatrix}$

$2A = \begin{bmatrix} -6 & 6 & 0 \\ 9 & 3 & 12 \end{bmatrix} + \begin{bmatrix} -10 & 0 & 10 \\ -35 & -5 & -30 \end{bmatrix}$

$2A = \begin{bmatrix} -6-10 & 6+0 & 0+10 \\ 9-35 & 3-5 & 12-30 \end{bmatrix}$

$2A = \begin{bmatrix} -16 & 6 & 10 \\ -26 & -2 & -18 \end{bmatrix}$

$A = \frac{1}{2}\begin{bmatrix} -16 & 6 & 10 \\ -26 & -2 & -18 \end{bmatrix}$

$A = \begin{bmatrix} -8 & 3 & 5 \\ -13 & -1 & -9 \end{bmatrix}$

1.8 Multiplication of Matrices:

Two matrices A and B are said to be conformable for the product AB if the number of columns of A is equal to the number of rows in B. That is, if A is of size m x n then B must be of size n x p , then product AB would be a matrix of size m x p defined by

$AB = (c_{ij})_{m \times p}$ where

$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$.

1.8.1 Properties of Matrix Multiplication:

(i) Matrix Multiplication is associative: If three matrices A, B and C are conformable for multiplication in the order ABC, then (AB)C = A(BC) = ABC.

(ii) Matrix Multiplication is distributive with respect to addition of matrices: A(B+C) = AB + AC holds good for the matrices A , B and C provided that they are conformable for the multiplication and the sum.

(iii) Matrix multiplication in general is non-commutative: $AB \neq BA$, Although both AB and BA may be defined.

(iv) If AB is a null matrix, that is AB = O, it does not necessarily mean that either A or B should be null matrix.

Example 5: Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 3 & 2 & -1 \end{bmatrix}$ $and\ B = \begin{bmatrix} 2 & 3 \\ -1 & 2 \\ 4 & -5 \end{bmatrix}$. Find AB and BA and show that $AB \neq BA$.

Solution : Here, A is a 2 × 3 matrix and B is a 3 × 2 matrix. Si, AB exist and it is of order 2 × 2 .

$$\therefore AB = \begin{bmatrix} 1 & -2 & 3 \\ 3 & 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 2 \\ 4 & -5 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2+2+12 & 3-4-15 \\ 6-2-4 & 9+4+5 \end{bmatrix}$$

$$AB = \begin{bmatrix} 16 & -16 \\ 0 & 18 \end{bmatrix}$$

Again, B is a 3 × 2 matrix and A is a 2 × 3 matrix. So, BA exist and it is of order 3 × 3.

$$\therefore BA = \begin{bmatrix} 2 & 3 \\ -1 & 2 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ 3 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 2+9 & -4+6 & 6-3 \\ -1+6 & 2+4 & -3-2 \\ 4-15 & -8-10 & 12+5 \end{bmatrix} = \begin{bmatrix} 11 & 2 & 3 \\ 5 & 6 & -5 \\ -11 & -18 & 17 \end{bmatrix}$$

$Clearly\ ,\ AB \neq BA.$

1.9 Positive Integral Powers of A Square Matrix:

For any square matrix, we define (i) $A^1 = A\ \ and\ , (ii) A^{n+1} = A^n . A, where\ n \in \mathbb{N}$.

It is evident from this definition that $A^2 = A.A, A^3 = A^2 A = (AA)A$ etc.

It can be easily shown that

(i) $A^m A^n = A^{m+n}$ $and , (ii) (A^m)^n = A^{mn}$ $for\ all\ m , n \in N$.

Matrix Polynomial : Let $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n$ be a polynomial and let A be a square matrix of order n. Then,

$f(A) = a_0 A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_{n-1} A + a_n I_n$

Is called a matrix polynomial.

For Example, if $f(x) = x^2 - 3x + 2$ is a polynomial and A is a square matrix, then $f(A) = A^2 - 3A + 2I$ is a matrix polynomial.

1.9.1 Type I On Multiplication Of Matrices:

Example 6: Find the value of x such that $\begin{bmatrix} \mathbf{1} & \boldsymbol{x} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{2} \\ \mathbf{2} & \mathbf{5} & \mathbf{1} \\ \mathbf{15} & \mathbf{3} & \mathbf{2} \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \boldsymbol{x} \end{bmatrix} = \mathbf{0}.$

Solution: We have, $\begin{bmatrix} 1 & x & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = 0$

$\Rightarrow \begin{bmatrix} 1 & x & 1 \end{bmatrix} \begin{bmatrix} 7 + 2x \\ 12 + x \\ 21 + 2x \end{bmatrix} = 0$

$\Rightarrow 7 + 2x + 12x + x^2 + 21 + 2x = 0$

$\Rightarrow x^2 + 16x + 28 = 0$

$\Rightarrow (x + 14)(x + 2) = 0$

$\Rightarrow x = -2\ or - 14$.

1.9.2 Type II On Matrix Polynomials And Matrix Polynomial Equations:

Example 7: Let, $\boldsymbol{f(x) = x^2 - 5x + 6}$. $\boldsymbol{Find\ f(A), if\ A} = \begin{bmatrix} \mathbf{2} & \mathbf{0} & \mathbf{1} \\ \mathbf{2} & \mathbf{1} & \mathbf{3} \\ \mathbf{1} & \mathbf{-1} & \mathbf{0} \end{bmatrix}$.

Solution : First we note that by f (A) we mean the matrix polynomial $A^2 - 5A + 6I_3$. That is, to obtain f (A) , x is replaced by A and the constant term is multiplied by the identity matrix of order same as that of A.

Now, $A^2 = AA = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 4+0+1 & 0+0-1 & 2+0+0 \\ 4+2+3 & 0+1-3 & 2+3+0 \\ 2-2+0 & 0-1+0 & 1-3+0 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$

$-5A = \begin{bmatrix} -10 & 0 & -5 \\ -10 & -5 & -15 \\ -5 & 5 & 0 \end{bmatrix}$ $and\ 6I_3 = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

$\therefore f(A) = A^2 - 5A + 6I_3 = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} + \begin{bmatrix} -10 & 0 & -5 \\ -10 & -5 & -15 \\ -5 & 5 & 0 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$.

1.9.3 Type III On Principle of Mathematical Induction

The Principle of Mathematical Induction:

Let P (n) be a statement involving positive integer n such that

(i) P(1) is true i.e.,the statement is true for n = 1, and

(ii) $P\ (m+1)$ is true whenever P(m) is true i.e, the truth of P(m) implies the truth of P(m+1).

Then, $P(n)$ is true for all positive integer n.

Example 8: If $A_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\ \alpha \end{bmatrix}$, then prove that (i) $A_\alpha A_\beta = A_{\alpha+\beta}$, (ii) $(A_\alpha)^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$ for every positive integer n.

Solution :

(i) We find that $A_\alpha A_\beta = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\ \alpha \end{bmatrix} \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}$

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & \sin\alpha\cos\beta + \cos\alpha\, sin\beta \\ -\sin\alpha\cos\beta - \cos\alpha\sin\beta & \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) \\ -\sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

$$= A_{\alpha+\beta}\ .$$

(ii) We shall prove the result by mathematical induction on n.

Step 1 When n = 1, by the definition of integral powers of a matrix, we obtain

$$(A_\alpha)^1 = A_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \cos(1.\alpha) & \sin(1.\alpha) \\ -\sin(1.\alpha) & \cos(1.\alpha) \end{bmatrix}$$

So, the result is true for n = 1.

Step 2 Let the result is true for n = m. Then,

$$(A_\alpha)^m = \begin{bmatrix} \cos m\alpha & \sin m\alpha \\ -\sin m\alpha & \cos m\alpha \end{bmatrix}$$

Now, we will show that the result is true for $n = m + 1$

$$i.e. (A_\alpha)^{m+1} = \begin{bmatrix} \cos(m+1)\alpha & \sin(m+1)\alpha \\ -\sin(m+1)\alpha & \cos(m+1)\alpha \end{bmatrix}$$

By the definition of integral powers of a square matrix, we have

$$(A_\alpha)^{m+1} = (A_\alpha)^m A_\alpha$$

$$(A_\alpha)^{m+1} = \begin{bmatrix} \cos m\alpha & \sin m\alpha \\ -\sin m\alpha & \cos m\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$(A_\alpha)^{m+1} = \begin{bmatrix} \cos m\alpha \cos\alpha - \sin m\alpha \sin\alpha & \cos m\alpha \sin\alpha + \sin m\alpha \cos\alpha \\ -\sin m\alpha \cos\alpha - \cos m\alpha \sin\alpha & -\sin m\alpha \sin\alpha + \cos m\alpha \cos\alpha \end{bmatrix}$$

$$(A_\alpha)^{m+1} = \begin{bmatrix} \cos(m\alpha+\alpha) & \sin(m\alpha+\alpha) \\ -\sin(m\alpha+\alpha) & \cos(m\alpha+\alpha) \end{bmatrix} = \begin{bmatrix} \cos(m+1)\alpha & \sin(m+1)\alpha \\ -\sin(m+1)\alpha & \cos(m+1)\alpha \end{bmatrix}$$

This shows that the result is true for n = m + 1, whenever it is true for n = m.

Hence, by the principle of mathematical induction, the result is valid for any positive integer n.

1.10 Transpose of a Matrix

Let $A = [a_{ij}]$ be an $m \times n$ matrix. Then, the transpose of A, denoted by $A^T or\ A', is\ an\ n \times m\ matrix$ such that $(A^T)_{ij} = a_{ij} \quad for\ all\ \ i = 1, 2, \ldots, n; j = 1, 2, \ldots, m.$

Thus, A^T is obtained from A by changing its rows into columns and columns into rows.

For example, if $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, then\ A^T = [1 \quad 2 \quad 3]$.

The first row of A^T is the first column of A. The second row of A^T is the second column of A and so on.

1.10.1 Properties of Transpose

Let A and B be two matrices, then

I. $(A')' = A$

II. $(A+B)' = A' + B'$; A and B being the same order.

III. $(kA)' = kA'$, k be any scalar.

IV. $(AB)' = B'A'$; A and B being conformable for the product AB.

1.11 Symmetric and Skew - symmetric Matrix:

A square matrix A is said to be symmetric if its transpose coincides with itself, i.e, $A^T = A$.

A square matrix A is said to be skew - symmetric if $A^T = -A$.

Theorem: Every square matrix can be uniquely expressed as the sum of a symmetric matrix and a skew- symmetric matrix.

Solution: Let A be any square matrix. Then we have

$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

Denoting $\frac{1}{2}(A + A^T)$ by P and $\frac{1}{2}(A - A^T)$ by Q , we have A = P + Q

Now, $P' = \{\frac{1}{2}(A + A^T)\}^T = \frac{1}{2}\{A^T + (A^T)^T\} = \frac{1}{2}\{A^T + A\} = \frac{1}{2}(A + A^T) = P$

Which follows that P is a symmetric matrix.

Also, $Q' = \{\frac{1}{2}(\mathrm{A} - \mathrm{A}^{\mathrm{T}})\}^T = \frac{1}{2}\{A^T - (A^T)^T\} = \frac{1}{2}\{A^T - A\} = -\frac{1}{2}(\mathrm{A} - \mathrm{A}^{\mathrm{T}}) = -\mathrm{Q}$

Which follows that Q is a skew-symmetric matrix.

Thus the square matrix A is expressible as the sum of a symmetric matrix P and a skew-symmetric matrix Q.

Example 9: A matrix which is both symmetric as well as skew-symmetric is a null matrix.

Solution:

Let $A = [a_{ij}]$ a matrix which is both symmetric and skew- symmetric.

Now, $A = [a_{ij}]$ is a symmetric matrix $\Rightarrow a_{ij} = a_{ji}\ for\ all\ i\ ,j$ ....(i)

Also, $A = [a_{ij}]$ is a skew- symmetric matrix.

$\therefore a_{ij} = -a_{ji}\ for\ all\ i\ ,j\ \Rightarrow\ a_{ji} = -a_{ij}\ for\ all\ i\ ,j$ ....(ii)

From (i) and (ii), we obtain

$a_{ij} = -a_{ij}\ \ for\ all\ i\ ,j\ \Rightarrow 2a_{ij} = 0\ \ for\ all\ i\ ,j\ \Rightarrow a_{ij} = 0\ for\ all\ i\ ,j\ \Rightarrow A = [a_{ij}]\ is\ a\ null\ matrix.$

Example 10: Express $\begin{bmatrix} -3 & 4 & 1 \\ 2 & 3 & 0 \\ 1 & 4 & 5 \end{bmatrix}$ as a sum of a symmetric matrix and a skew- symmetric matrix.

Solution : We know that $\mathrm{A} = \frac{1}{2}(\mathrm{A} + \mathrm{A}^{\mathrm{T}}) + \frac{1}{2}(\mathrm{A} - \mathrm{A}^{\mathrm{T}})$ where $\frac{1}{2}(\mathrm{A} + \mathrm{A}^{\mathrm{T}})$ is symmetric and $\frac{1}{2}(\mathrm{A} - \mathrm{A}^{\mathrm{T}})$ is skew-symmetric.

Here, $\mathrm{A} = \begin{bmatrix} -3 & 4 & 1 \\ 2 & 3 & 0 \\ 1 & 4 & 5 \end{bmatrix}$

$\mathrm{A}^{\mathrm{T}} = \begin{bmatrix} -3 & 2 & 1 \\ 4 & 3 & 4 \\ 1 & 0 & 5 \end{bmatrix}$

$\therefore \frac{1}{2}(\mathrm{A} + \mathrm{A}^{\mathrm{T}}) = \frac{1}{2}\begin{bmatrix} -3-3 & 4+2 & 1+1 \\ 2+4 & 3+3 & 0+4 \\ 1+1 & 4+0 & 5+5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -6 & 6 & 2 \\ 6 & 6 & 4 \\ 2 & 4 & 10 \end{bmatrix} = \begin{bmatrix} -3 & 3 & 1 \\ 3 & 3 & 2 \\ 1 & 2 & 5 \end{bmatrix}$

And $\frac{1}{2}(\mathrm{A} - \mathrm{A}^{\mathrm{T}}) = \frac{1}{2}\begin{bmatrix} -3+3 & 4-2 & 1-1 \\ 2-4 & 3-3 & 0-4 \\ 1-1 & 4-0 & 5-5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -4 \\ 0 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix}$

$\mathrm{A} = \begin{bmatrix} -3 & 3 & 1 \\ 3 & 3 & 2 \\ 1 & 2 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix}$

1.12 Some typical type of Matrices

1. Idempotent Matrix

A square matrix A is said to be idempotent matrix if $A^2 = A$.

Example : matrix A = $\begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$ is idempotent, for

$$A^2 = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix} \times \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix} = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix} = A .$$

2. Nilpotent Matrix

A square matrix A is said to be nilpotent matrix of index k, if k be the least positive integer for which $A^k = 0$, null matrix.

For Example : $A = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 3 & -3 \\ -4 & 4 & -4 \end{bmatrix}$

$$\text{Then } A^2 = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 3 & -3 \\ -4 & 4 & -4 \end{bmatrix} \times \begin{bmatrix} 1 & -1 & 1 \\ -3 & 3 & -3 \\ -4 & 4 & -4 \end{bmatrix} = \begin{bmatrix} 1+3-4 & -1-3+4 & 1+3-4 \\ -3-9+12 & 3+9-12 & -3-9+12 \\ -4-12+16 & 4+12-16 & -4-12+16 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

There fore, A is a nilpotent matrix of index 2.

3. Involutary Matrix :

A square matrix A is said to be involutary matrix if $A^2 = I$

For Example : If A = $\begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix} \times \begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 16-3-12 & 12+0-12 & 12-3-9 \\ -4+0+4 & -3+0+4 & -3+0+3 \\ -16+4+12 & -12+0+12 & -12+4+9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Therefore, A is an involutary matrix.

4. Orthogonal Matrix:

A square matrix A is said to be orthogonal if $AA' = A'A = I$.

Properties of Orthogonal Matrices

1. If A is orthogonal, then $A^{-1} and\ A'$ are also orthogonal.
2. If A and B are orthogonal, then AB is also orthogonal.

Example 11: Show that the matrix $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.

Solution : To prove A is orthogonal, we have to show that $A'A = I$.

$A' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$\therefore AA' = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \times \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$= \begin{bmatrix} cos^2\theta + sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & sin^2\theta + cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Example 12: If A = $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, prove that $A^2 - 4A - 5I = 0$.

Solution : $A^2 = A \times A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1\times1+2\times2+2\times2 & 1\times2+2\times1+2\times2 & 1\times2+2\times2+2\times1 \\ 2\times1+1\times2+2\times2 & 2\times2+1\times1+2\times2 & 2\times2+1\times2+2\times1 \\ 2\times1+2\times2+1\times2 & 2\times2+2\times1+1\times2 & 2\times2+2\times2+1\times1 \end{bmatrix}$

$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$

$A^2 - 4A - 5I = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$= \begin{bmatrix} 9-4-5 & 8-8-0 & 8-8-0 \\ 8-8-0 & 9-4-5 & 8-8 \\ 8-8-0 & 8-8-0 & 9-4-5 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$= O$.

Example 13: If A is a symmetric matrix and B is skew- symmetric matrix such that $A + B = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix}$, then find AB.

Solution: It is given that A is a symmetric matrix and B is a skew-symmetric matrix.

Therefore, $A^T = A \ and \ B^T = -B$.

Now, $A + B = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix}$ ………….(i)

$\Rightarrow (A+B)^T = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix}^T \Rightarrow A^T + B^T = \begin{bmatrix} 2 & 5 \\ 3 & -1 \end{bmatrix} \Rightarrow A - B = \begin{bmatrix} 2 & 5 \\ 3 & -1 \end{bmatrix} \dots\dots\dots\dots..(ii)$

Adding (i) and (ii) , we obtain

$$2A = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 5 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 2+2 & 3+5 \\ 5+3 & -1-1 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 8 & -2 \end{bmatrix}$$

$$\Rightarrow A = \frac{1}{2}\begin{bmatrix} 4 & 8 \\ 8 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & -1 \end{bmatrix} \dots\dots\dots\dots (iii)$$

From (i) and (iii), we obtain

$$\begin{bmatrix} 2 & 4 \\ 4 & -1 \end{bmatrix} + B = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix} \Rightarrow B = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\therefore AB = \begin{bmatrix} 2 & 4 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0+4 & -2+0 \\ 0-1 & -4+0 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -1 & -4 \end{bmatrix}$$

Example 14: Find the values of x , y, z if the matrix $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ satisfy the equation $A^T A = I_3$.

Solution: We have,

$$A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix}$$

It is given that

$$A^T A = I_3$$

$$\Rightarrow \begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix}\begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x^2 & 0 & 0 \\ 0 & 6y^2 & 0 \\ 0 & 0 & 3z^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow 2x^2 = 1, 6y^2 = 1 , 3z^2 = 1$$

$$\Rightarrow x = \pm\frac{1}{\sqrt{2}} , \ \ y = \pm\frac{1}{\sqrt{6}} , \ \ z = \pm\frac{1}{\sqrt{3}}$$

Example 15: If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is a matrix satisfying $AA^T = 9I_3$, then find the values of a and b.

Solution: We have,

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix}$$

$\therefore AA^T = 9I_3$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix} \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix} = 9 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 9 & 0 & a+2b+4 \\ 0 & 9 & 2a+2-2b \\ a+2b+4 & 2a+2-2b & a^2+4+b^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$\Rightarrow a + 2b + 4 = 0\,,\ 2a + 2 - 2b = 0\ and\ a^2 + 4 + b^2 = 9$

$\Rightarrow a + 2b + 4 = 0\,, a - b + 1 = 0\ and\ \ a^2 + b^2 = 5$

Solving $a + 2b + 4 = 0\ \ and\ a - b + 1 = 0,\ \ we\ get:\ a = -2\ \ and\ b = -1.$

Exercise 1

❖ **MCQ :**

1. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}. A^2 = ?$

   a. A , b. 3A , c. unit matrix, d. 2A

2. If the matrix $\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ \lambda & -3 & 0 \end{bmatrix}$ is singular, the value of λ is –
   a) 0
   b) 4
   c) 2
   d) -1

3. If the matrix $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & x \end{bmatrix}$ is singular then the value of x is

   a. 3, b. 5, c. 2, d. 4

4. If $B^2 = I\ and\ A = I - B$ then

   a. $A^2 = I$, b. BA = 0 , c. $A^2 = A$, d. AB = 0

5. If A = $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ then $A^{100} = ?$

   a. $2^{99} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ b. $2^{101} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ c. $2^{100} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ d. none

6. Which is true about matrix multiplication

a. it is associative. b. it is commutative.

c. it is both associative and commutative d. none.

7. The matrix $\begin{bmatrix} 0 & 5 & -7 \\ -5 & 0 & 11 \\ 7 & -11 & 0 \end{bmatrix}$ is known as

a. symmetric matrix, c. diagonal matrix

b. skew-symmetric matrix, d. scaler matrix.

8. If $\begin{bmatrix} 3 & -2 \\ 5 & 6 \end{bmatrix} + 2A = \begin{bmatrix} 5 & 6 \\ -7 & 10 \end{bmatrix}, then\ A = ?$
   a) $\begin{bmatrix} 1 & 3 \\ -5 & 4 \end{bmatrix}$,
   b) $\begin{bmatrix} -15 & 0 \\ -3 & 4 \end{bmatrix}$
   c) $\begin{bmatrix} 1 & 4 \\ -6 & 2 \end{bmatrix}$,
   d) $none\ of\ these.$
9. If $\begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$ $, then$
   a) $x = -2, y = 8;$
   b) $x = 2, y = -8;$
   c) $x = 3, y = -6$;
   d) $x = -3$, $y = 6;$
10. If $\begin{bmatrix} x & y \\ 3y & x \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, then
   a) $x = 1, y = 2$;
   b) $x = 2$, $y = 1$;
   c) $x = 1$, $y = 1;$
   d) $none\ of\ these.$
11. If $A = \begin{bmatrix} 3 - 2x & x + 1 \\ 2 & 4 \end{bmatrix}$ is a singular matrix, then x = ?
   a) 0 ;
   b) 1 ;
   c) -1 ;
   d) -2 ;
12. Let A be an $m \times n$ matrix and B be $p \times q$ matrix, then AB is defined if
   a) $n = p$
   b) $m = p$
   c) $m = q$
   d) $p = q$
13. A square matrix A is said to be singular if
   a) detA = -1
   b) detA = 0
   c) detA = 1
   d) detA = -2

14. The value k for which the matrix $\begin{bmatrix} 2 & 0 & 1 \\ 1 & -k & 2 \\ 3 & 1 & 0 \end{bmatrix}$ is singular is –

a) 1
b) 2
c) 3
d) 4

15. The matrix $A = \begin{bmatrix} ab & -b^2 \\ -a^2 & ab \end{bmatrix}$ is

a) Idempotent
b) Orthogonal
c) Nilpotent
d) None of these

Answers : 1.b 2.c 3.a 4.b 5.a 6.a, 7.d 8.b 9. c

10. c, 11.d, 12.c, 13.b , 14.c , 15.a.

- **Short Answer Type Question:**

1. If $\begin{bmatrix} y & 1 \\ 3 & x \end{bmatrix} + \begin{bmatrix} x & 1 \\ -1 & -y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ then find the values of x and y.
2. Find the value of t for which the matrix $\begin{bmatrix} 2 & 0 & 1 \\ 5 & t & 3 \\ 0 & 3 & 1 \end{bmatrix}$ is singular.
3. If $A = (2 \quad -3 \quad 4)$, $B = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, $C = (0 \quad 2 \quad 3)$ and $D = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$ then find AB+CD
4. If $2X + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix}$. Find X
5. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, Show that $A^2 - 5A + 7I = O$. Use this to find A^4.
6. If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$, *find k such that* $A^2 = kA - 2I_2$.
7. If $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$, find k such that $A^2 - 8A + kI = O$.
8. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $f(x) = x^2 - 2x - 3$, *show that* $f(A) = O$.
9. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ *and* $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, *then find* λ, μ *so that* $A^2 = \lambda A + \mu I$.
10. Find the value of x for which the matrix product $\begin{bmatrix} 2 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} -x & 14x & 7x \\ 0 & 1 & 0 \\ x & -4x & -2x \end{bmatrix}$ equal to an identity matrix.
11. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$, *then show that* $A^3 - 23A - 40I = O$
12. If $(x) = x^2 - 2x$, find f(A) , where $A = \begin{bmatrix} 0 & 1 & 2 \\ 4 & 5 & 6 \\ 0 & 2 & 3 \end{bmatrix}$.

13. Let $A = \begin{bmatrix} 2 & -3 \\ -7 & 5 \end{bmatrix}$ $and\ B = \begin{bmatrix} 1 & 0 \\ 2 & -4 \end{bmatrix}$, Verify that (i) $(2A)^T = 2A^T$, (ii) $(A+B)^T = A^T + B^T$, (iii) $(A-B)^T = A^T - B^T$, (iv) $(AB)^T = B^T A^T$

14. If $A = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$ $and\ B = \begin{bmatrix} 1 & 0 & 4 \end{bmatrix}, verify\ that\ (AB)^T = B^T A^T.$

15. For two matrices A and B, $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 0 \end{bmatrix}$ $, B = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 5 & 0 \end{bmatrix}$ $Verify\ that\ (AB)^T = B^T A^T.$

16. If $A^T = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$ $and\ B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ $, find\ A^T - B^T.$

17. If $\begin{bmatrix} 4 & x+2 \\ 2x-3 & x+1 \end{bmatrix}$ is a symmetric matrix, then find the value of x .

18. If the matrix $A = \begin{bmatrix} 5 & 2 & x \\ y & z & -3 \\ 4 & t & -7 \end{bmatrix}$ is a symmetric matrix, find x , y ,z , t .

19. Express the matrix $\begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$ as the sum of a symmetric and skew- symmetric matrix and verify your result.

20. Let , $A = \begin{bmatrix} 3 & 2 & 7 \\ 1 & 4 & 3 \\ -2 & 5 & 8 \end{bmatrix}$, Find matrices X and Y such that $X + Y = A$, where X is a symmetric and Y is a skew- symmetric matrix.

Solution :

1. $x = 1, y = 0$;
2. $t = \frac{3}{2}$;
3. 20,
4. $X = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$
5. $\begin{bmatrix} 39 & 55 \\ -55 & -16 \end{bmatrix}$
6. K = 1
7. K = 7

9. $\lambda = 4, \mu = -1$

10. 1/5 ;

12. $\begin{bmatrix} 4 & 7 & 2 \\ 12 & 19 & 8 \\ 8 & 12 & 3 \end{bmatrix}$

16. $\begin{bmatrix} 4 & 3 \\ -3 & 0 \\ -1 & -2 \end{bmatrix}$

17. 5

18. $x = 4, y = 2, z \in C, t = -3$

19. Symmetric matrix = $\begin{bmatrix} 3 & \frac{1}{2} & -\frac{5}{2} \\ \frac{1}{2} & -2 & -2 \\ -\frac{5}{2} & -2 & 2 \end{bmatrix}$; skew-symmetric matrix = $\begin{bmatrix} 0 & -\frac{5}{2} & -\frac{3}{2} \\ \frac{5}{2} & 0 & -3 \\ \frac{3}{2} & 3 & 0 \end{bmatrix}$.

20. $X = \begin{bmatrix} 3 & 3/2 & 5/2 \\ 3/2 & 4 & 4 \\ 5/2 & 4 & 8 \end{bmatrix}, Y = \begin{bmatrix} 0 & 1/2 & 9/2 \\ -1/2 & 0 & -1 \\ -9/2 & 1 & 0 \end{bmatrix}$

2 Long Answer Type Question:

1. Find X and Y if $X + Y = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}$, $X - Y = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 2 & 0 \\ -1 & -1 & 2 \end{bmatrix}$.

   Ans. $X = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, Y = \begin{bmatrix} 0 & -2 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

2. Compute AB and BA and show that $AB \neq BA$.

   $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}, and\ B = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

3. If $A = \begin{bmatrix} 2 & -3 & 1 \\ 2 & -1 & 1 \\ 2 & -1 & 1 \end{bmatrix}\ and\ B = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 3 & -3 \end{bmatrix}, prove\ that\ AB = 0.$

4. Find x if $[1 \quad 1 \quad x] \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0.$ (Ans. $x = -2$)

5. Verify that $(AB)' = B'A'$ where,

   $A = \begin{bmatrix} 1 & 4 \\ 0 & 5 \\ 6 & 7 \end{bmatrix}\ and\ B = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & -7 \end{bmatrix}.$

CHAPTER 2

Determinants

2.1 Introduction

In Engineering Mathematics, solution of simultaneous equations is very important. In this chapter we shall study the system of linear equations with emphasis on their solution by means of determinants.

2.2 Determinant

The notation of determinants arises from the process of elimination of the unknowns of simultaneous linear equations.

Consider the two linear equations in x,

$$a_1 x + b_1 = 0 \quad \text{... (1)}$$

$$a_2 x + b_2 = 0 \quad \text{... (2)}$$

From (1) $$x = -\frac{b_1}{a_1}$$

Substituting the value of x in (2); we get the eliminant

$$a_2\left(-\frac{b_1}{a_1}\right) + b_2 = 0$$

or $$a_1 b_2 - a_2 b_1 = 0. \quad \text{... (3)}$$

From (1) and (2) by suppressing x, the eliminant is written as

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0 \quad \text{... (4)}$$

when the two **rows** of a_1, b_1 and a_2, b_2 are enclosed by two vertical bars then it is called a **determinant** of **second order.**

$$\begin{vmatrix} a_1 \\ a_2 \end{vmatrix} \; and \; \begin{vmatrix} b_1 \\ b_2 \end{vmatrix}$$

Column 1 Column 2

Row 1 ⟶ $a_1 \ldots\ldots b_1$

Row 2 ⟶ $a_2 \ldots\ldots b_2$

Each quantity a_1, b_1, a_2, b_2 is called an **element** or a constituent of the determinant. From (3) and (4), we know that both expressions are eliminant, so we equate them.

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \quad \text{or} \quad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

$a_1b_2 - a_2b_1$ is called the expansion of the determinant of $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.

Example 1. *Expand the determinant* $\begin{vmatrix} 3 & 2 \\ 6 & 7 \end{vmatrix}$.

Solution. $\begin{vmatrix} \overset{+}{3} & \overset{-}{2} \\ 6 & 7 \end{vmatrix} = (3) \times (7) - (2) \times (6) = 21 - 12 = 9.$ **Ans.**

Exercise 2.1

Expand the following determinants :

1. $\begin{vmatrix} 4 & 6 \\ 2 & 5 \end{vmatrix}$ **Ans.** 8

2. $\begin{vmatrix} -3 & 7 \\ 2 & 4 \end{vmatrix}$ **Ans.** – 26

3. $\begin{vmatrix} 8 & 5 \\ 3 & 1 \end{vmatrix}$ **Ans.** – 7

4. $\begin{vmatrix} 5 & -2 \\ 4 & 3 \end{vmatrix}$ **Ans.** 23

2.3 Determinant as eliminant

Consider the following three equations having three unknowns, x, y and z.

$$a_1x + b_1y + c_1z = 0 \quad \text{...(1)}$$
$$a_2x + b_2y + c_2z = 0 \quad \text{...(2)}$$
$$a_3x + b_3y + c_3z = 0 \quad \text{...(3)}$$

From (2) and (3) by cross-multiplication, we get

$$\frac{x}{b_2c_3 - b_3c_2} = \frac{y}{a_3c_2 - a_2c_3} = \frac{z}{a_2b_3 - a_3b_2} = k \ (say)$$

$$x = (b_2c_3 - b_3c_2)\,k$$
$$y = (a_3c_2 - a_2c_3)\,k$$

and $z = (a_2b_3 - a_3b_2)\,k$

Substituting the values of x, y and z in (1), we get the eliminant

$$a_1(b_2c_3 - b_3c_2)\,k + b_1(a_3c_2 - a_2c_3)\,k + c_1(a_2b_3 - a_3b_2)\,k = 0$$

or $a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) = 0$ ...(4)

From (1), (2) and (3) by suppressing x, y, z the remaining can be written in the determinant as

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \quad \text{...(5)}$$

This is determinant of third order.

As (4) and (5) both are the eliminant of the same equations.

$$\therefore \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) = 0.$$

2.4 Minor

The minor of an element is defined as a determinant obtained by deleting the row and column containing the element.

Thus the minors a_1, b_1 and c_1 are respectively: $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, $\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$ and $\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$.

Thus $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1$ (minor of a_1) – b_1 (minor of b_1) + c_1 (minor of c_1).

2.5 Cofactor

Cofactor = $(-1)^{r+c}$ Minor

where r is the number of rows of the element and c is the number of columns of the element.

The cofactor of any element of jth row and ith column is

$$(-1)^{i+j} \text{ minor}$$

Thus the cofactor of $a_1 = (-1)^{1+1}(b_2c_3 - b_3c_2) = +(b_2c_3 - b_3c_2)$

The cofactor of $b_1 = (-1)^{1+2}(a_2c_3 - a_3c_2) = -(a_2c_3 - a_3c_2)$

The cofactor of $c_1 = (-1)^{1+3}(a_2b_3 - a_3b_2) = +(a_2b_3 - a_3b_2)$

The determinant $= a_1$ (cofactor of a_1) + a_2 (cofactor of a_2) + a_3 (cofactor of a_3).

Example 2. *Find* :

(i) Minors (*ii*) *Cofactors of the elements of the first row of the determinant*

$$\begin{vmatrix} 2 & 3 & 5 \\ 4 & 1 & 0 \\ 6 & 2 & 7 \end{vmatrix}$$

Solution.

(*i*) The minor of the element (2) is

$$\begin{vmatrix} 2 & 3 & 5 \\ 4 & 1 & 0 \\ 6 & 2 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 2 & 7 \end{vmatrix} = (1)\times(7) - (0)\times(2) = 7 - 0 = 7$$

The minor of the element (3) is

$$\begin{vmatrix} 2 & 3 & 5 \\ 4 & 1 & 0 \\ 6 & 2 & 7 \end{vmatrix} = \begin{vmatrix} 4 & 0 \\ 6 & 7 \end{vmatrix} = (4)\times(7) - (0)\times(6) = 28 - 0 = 28$$

The minor of the element (5) is

$$\begin{vmatrix} 2 & 3 & 5 \\ 4 & 1 & 0 \\ 6 & 2 & 7 \end{vmatrix} = \begin{vmatrix} 4 & 1 \\ 6 & 2 \end{vmatrix} = (4)\times(2) - (1)\times(6) = 8 - 6 = 2$$

The cofactor of (2) = $(-1)^{1+1}(7) = +7$

The cofactor of (3) = $(-1)^{1+2}(28) = -28$ **Ans.**

The cofactor of (5) = $(-1)^{1+3}(2) = +2$.

Example 3. *Expand the determinant* $\begin{vmatrix} 6 & 2 & 3 \\ 2 & 3 & 5 \\ 4 & 2 & 1 \end{vmatrix}$

Solution. $\begin{vmatrix} 6 & 2 & 3 \\ 2 & 3 & 5 \\ 4 & 2 & 1 \end{vmatrix}$ = 6 (cofactor of 6) + 2 (cofactor of 2) + 3 (cofactor of 3).

$= 6(3 \times 1 - 5 \times 2) - 2(2 \times 1 - 4 \times 5) + 3(2 \times 2 - 3 \times 4) = -30.$ **Ans.**

Example 4. *Evaluate the determinant*

$$\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$$

(i) *With the help of second row,* (*ii*) *with the help of third column.*

Solution.

(*i*) $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$ = 3 × (cofactor of 3) + 5 × (cofactor of 5) + (– 1) (cofactor of – 1)

=-3(0-4)+5(2-0)+(1-0)=23.

Ans.

(*ii*) $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$ = 4 × (cofactor of 4) + (– 1) (cofactor of (– 1)) + 2 × (cofactor of 2)

$$= 4 \times (-1)^{1+3} \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} + (-1)(-1)^{2+3} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + 2 \times (-1)^{3+3} \begin{vmatrix} 1 & 0 \\ 3 & 5 \end{vmatrix}$$

= 23.

Example 5. *Expand the fourth order determinant* $\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 2 & 0 \\ 2 & 0 & 1 & 3 \\ 1 & 2 & 1 & 0 \end{vmatrix}$

Solution. Given determinant $= (0)(-1)^{1+1} \begin{vmatrix} 0 & 2 & 0 \\ 0 & 1 & 3 \\ 2 & 1 & 0 \end{vmatrix} + 1(-1)^{1+2} \begin{vmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 1 & 1 & 0 \end{vmatrix}$

$$+ 2(-1)^{1+3} \begin{vmatrix} 1 & 0 & 0 \\ 2 & 0 & 3 \\ 1 & 2 & 0 \end{vmatrix} + 3(-1)^{1+4} \begin{vmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= 0 - \begin{vmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 1 & 1 & 0 \end{vmatrix} + 2 \begin{vmatrix} 1 & 0 & 0 \\ 2 & 0 & 3 \\ 1 & 2 & 0 \end{vmatrix} - 3 \begin{vmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 1 \end{vmatrix}$$

= -3-12-18

= -33.

Therefore,

$$\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 2 & 0 \\ 2 & 0 & 1 & 3 \\ 1 & 2 & 1 & 0 \end{vmatrix} = -33$$

Ans.

Exercise 2.2

Write the minors and co factors of each element of the following determinants and also evaluate the determinant in each case :

1. $\begin{vmatrix} -2 & 3 \\ 4 & -9 \end{vmatrix}$ $M_{11} = -9,\ M_{12} = 4,\ M_{21} = 3,\ M_{22} = -2$
$A_{11} = -9,\ A_{12} = -4,\ A_{21} = -3,\ A_{22} = -2 \quad |A| = 6$ **Ans.**

2. $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$ $M_{11} = \cos\theta,\ M_{12} = \sin\theta,\ M_{12} = -\sin\theta, M_{22} = \cos\theta$
$A_{11} = \cos\theta, A_{12} = -\sin\theta, A_{21} = \sin\theta,\ A_{22} = \cos\theta, \quad |A| = 1$ **Ans.**

3. $\begin{vmatrix} 42 & 1 & 6 \\ 28 & 7 & 4 \\ 14 & 3 & 2 \end{vmatrix}$ $M_{11} = 2,\ M_{12} = 0,\ M_{13} = -14,\ M_{21} = -16,\ M_{22} = 0$
$M_{23} = 112,\ M_{31} = -38, M_{32} = 0,\ M_{33} = 266$
$A_{11} = 2,\ A_{12} = 0,\ A_{13} = -14,\ A_{21} = 16,\ A_{22} = 0$
$A_{23} = -112,\ A_{31} = -38,\ A_{32} = 0,\ A_{33} = 266, \quad |A| = 0$ **Ans.**

4. $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ $M_{11} = (ab^2 - ac^2),\ M_{12} = (ab - ac),\ M_{13} = (c - b),\ M_{21} = a^2b - bc^2$
$M_{22} = (ab - bc),\ M_{23} = (c - a),\ M_{31} = (ca^2 - cb^2),\ M_{32} = ca - bc, M_{33} = (b - a),$
$A_{11} = (ab^2 - ac^2),\ A_{12} = (ac - ab), A_{13} = (c - b), A_{21} = bc^2 - a^2b$
$A_{22} = (ab - bc),\ A_{23} = (a - c), A_{31} = (ca^2 - cb^2),\ A_{32} = (bc - ca), A_{33} = (b - a)$
$|A| = (a - b)(b - c)(c - a).$ **Ans.**

Expand the following determinants :

5. $\begin{vmatrix} 2 & -3 & 4 \\ 5 & 1 & -6 \\ -7 & 8 & -9 \end{vmatrix}$ **6.** $\begin{vmatrix} 5 & 0 & 7 \\ 8 & -6 & -4 \\ 2 & 3 & 9 \end{vmatrix}$ **7.** $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

Ans. $|A| = 5$ **Ans.** $|A| = 42$ **Ans.** $|A| = abc + 2fgh - af^2 - bg^2 - ch^2$

Expand the following determinants by two methods :

(i) along the-third row.

(ii) along the-third column.

8. $\begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}$ **9.** $\begin{vmatrix} 3 & -2 & 4 \\ 1 & 2 & 1 \\ 0 & 1 & -1 \end{vmatrix}$ **10.** $\begin{vmatrix} 2 & 3 & -2 \\ 1 & 2 & 3 \\ -2 & 1 & -3 \end{vmatrix}$

Ans. $|A| = 40$ **Ans.** $|A| = -7$ **Ans.** $|A| = -37$

11. $\begin{vmatrix} \log_3 512 & \log_4 3 \\ \log_3 8 & \log_4 9 \end{vmatrix}$ **Ans.** $|A| = \frac{15}{2}$

12. If a, b, c are all positive and are the pth, qth, rth terms of a $G.P.$ respectively; then prove that

$$\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = 0$$

13. $\begin{vmatrix} 3 & 2 & 5 & 7 \\ -1 & -4 & -3 & 0 \\ 6 & 4 & 2 & -1 \\ 2 & -1 & 0 & 3 \end{vmatrix}$ **Ans.** 96

2.6 Rules of sarrus (For third order determinants only).

After writing the determinant, repeat the first two columns as below

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \left|\begin{array}{ccc|cc} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{array}\right.$$

(signs: $+ \quad + \quad + \quad - \quad - \quad -$)

$$= (a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3) + (-\,c_1b_2a_3 - a_1c_2b_3 - b_1a_2c_3)$$

Example 6. *Expand the determinant*

$$\Delta = \begin{vmatrix} 2 & 3 & 4 \\ 1 & 5 & 3 \\ 3 & 0 & 5 \end{vmatrix} \text{ by Rule of Sarrus.}$$

Solution.

$$\Delta = \left|\begin{array}{ccc|cc} 2 & 3 & 4 & 2 & 3 \\ 1 & 5 & 3 & 1 & 5 \\ 3 & 0 & 5 & 3 & 0 \end{array}\right.$$

(signs: $+ \quad + \quad + \quad - \quad - \quad -$)

$$=(2) \times (5) \times (5) + (3) \times (3) \times (3) + (4) \times (1) \times (0) - (4) \times (5) \times (3) - (2) \times (3) \times (0) - (3) \times (1) \times (5)$$

$$= 50 + 27 + 0 - 60 - 0 - 15 = 2$$ **Ans.**

Exercise 2.3

Expand the following determinants by Rule of Sarrus.

1. $\begin{vmatrix} 3 & 2 & -4 \\ 5 & 1 & -1 \\ -2 & 6 & 7 \end{vmatrix}$ **Ans.** – 155

2. $\begin{vmatrix} 1 & 4 & 2 \\ 2 & 5 & 3 \\ 3 & 6 & 4 \end{vmatrix}$ **Ans.** 0

3. $\begin{vmatrix} 6 & 3 & 7 \\ 32 & 13 & 37 \\ 10 & 4 & 11 \end{vmatrix}$ **Ans.** 10

4. $\begin{vmatrix} 9 & 25 & 6 \\ 7 & 13 & 5 \\ 9 & 23 & 6 \end{vmatrix}$ **Ans.** 6

5. If $a + b + c = 0$, solve the equation $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$

Ans. $x = \pm\sqrt{(a^2+b^2+c^2-ab-bc-ca)}$, $x = 0$

2.7 Properties of determinants

Property (*i*) The value of a determinant remains unaltered, if the rows are interchanged into columns (or the columns into rows).

Consider the determinant.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= a_1 (b_2c_3 - b_3c_2) - b_1 (a_2c_3 - a_3c_2) + c_1 (a_2b_3 - a_3b_2)$$

$$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1$$

$$= (a_1b_2c_3 - a_1b_3c_2) - (a_2b_1c_3 - a_2b_3c_1) + (a_3b_1c_2 - a_3b_2c_1)$$

$$= a_1 (b_2c_3 - b_3c_2) - a_2 (b_1c_3 - b_3c_1) + a_3 (b_1c_2 - b_2c_1)$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Proved.

Property (*ii*) If two rows (or two columns) of a determinant are interchanged, the sign of the value of the determinant changes.

Interchanging the first two rows of Δ, we get

$$\Delta' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= a_2 (b_1c_3 - b_3c_1) - b_2 (a_1c_3 - a_3c_1) + c_2 (a_1b_3 - a_3b_1)$$

$$= a_2b_1c_3 - a_2b_3c_1 - a_1b_2c_3 + a_3b_2c_1 + a_1b_3c_2 - a_3b_1c_2$$

$$= -[(a_1b_2c_3 - a_1b_3c_2) - (a_2b_1c_3 - a_3b_1c_2) + (a_2b_3c_1 - a_3b_2c_1)]$$

$$= -[(a_1 (b_2c_3 - b_3c_2) - b_1 (a_2c_3 - a_3c_2) + c_1 (a_2b_3 - a_3b_2)]$$

$$= -\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\Delta$$

Proved.

Property (*iii*) If two rows (or columns) of a determinant are identical, the value of the determinant is zero.

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$, so that the first two rows are identical.

By interchanging the first two rows, we get the same determinant Δ.

By property (*ii*), on interchanging the rows, the sign of the determinant changes.

or $\Delta = -\Delta$ or $2\Delta = 0$ or $\Delta = 0$ **Proved.**

Property (*iv*) If the elements of any row (or column) of a determinant be each multiplied by the same number, the determinant is multiplied by that number.

$$\Delta' = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= ka_1\,(b_2c_3 - b_3c_2) - kb_1\,(a_2c_3 - a_3c_2) + kc_1\,(a_2b_3 - a_3b_2)$$
$$= k\,[a_1\,(b_2c_3 - b_3c_2) - b_1\,(a_2c_3 - a_3c_2) + c_1\,(a_2b_3 - a_3b_2)]$$
$$= k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k\ \Delta.$$

Example 7. *Prove that*

$$\begin{vmatrix} a^2 & a & bc \\ b^2 & b & ca \\ c^2 & c & ab \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$$

Solution. $\begin{vmatrix} a^2 & a & bc \\ b^2 & b & ca \\ c^2 & c & ab \end{vmatrix}$

By multiplying R_1, R_2, R_3 by a, b and c respectively we get

$$= \frac{1}{abc}\begin{vmatrix} a^3 & a^2 & abc \\ b^3 & b^2 & abc \\ c^3 & c^2 & abc \end{vmatrix} = \frac{abc}{abc}\begin{vmatrix} a^3 & a^2 & 1 \\ b^3 & b^2 & 1 \\ c^3 & c^2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a^3 & a^2 & 1 \\ b^3 & b^2 & 1 \\ c^3 & c^2 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$$ By changing rows into columns

Proved

Example 8. *Without expanding and or evaluating, show that*

$$\begin{vmatrix} a^2 & a & 1 & bcd \\ b^2 & b & 1 & cda \\ c^2 & c & 1 & dab \\ d^2 & d & 1 & abc \end{vmatrix} = \begin{vmatrix} a^3 & a^2 & a & 1 \\ b^3 & b^2 & b & 1 \\ c^3 & c^2 & c & 1 \\ d^3 & d^2 & d & 1 \end{vmatrix}$$

Solution.

$$\begin{vmatrix} a^2 & a & 1 & bcd \\ b^2 & b & 1 & cda \\ c^2 & c & 1 & dab \\ d^2 & d & 1 & abc \end{vmatrix} = \frac{1}{abcd}\begin{vmatrix} a^3 & a^2 & a & abcd \\ b^3 & b^2 & b & abcd \\ c^3 & c^2 & c & abcd \\ d^3 & d^2 & d & abcd \end{vmatrix} \quad \begin{matrix} R_1 \to aR_1 \\ R_2 \to bR_2 \\ R_3 \to cR_3 \\ R_4 \to dR_4 \end{matrix}$$

$$= \frac{abcd}{abcd}\begin{vmatrix} a^3 & a^2 & a & 1 \\ b^3 & b^2 & b & 1 \\ c^3 & c^2 & c & 1 \\ d^3 & d^2 & d & 1 \end{vmatrix} C_4 \to \frac{1}{abcd} C_4 \quad = \begin{vmatrix} a^3 & a^2 & a & 1 \\ b^3 & b^2 & b & 1 \\ c^3 & c^2 & c & 1 \\ d^3 & d^2 & d & 1 \end{vmatrix}$$

Proved

Example 9. *Prove that* $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ *(Try yourself)*

Property (*v*) The value of the determinant remains unaltered if to the elements of one row (or column) be added any constant multiple of the corresponding elements of any other row (or column) respectively.

Let $$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

On multiplying the second column by l and the third column by m and adding to the first column we get

$$\Delta' = \begin{vmatrix} a_1 + lb_1 + mc_1 & b_1 & c_1 \\ a_2 + lb_2 + mc_2 & b_2 & c_2 \\ a_3 + lb_3 + mc_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + l\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} + m\begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix}$$

$= \Delta + 0 + 0$ (Since columns are identical)

$= \Delta$ **Proved**

Example 10. *Without expanding evaluate the determinant* $\Delta = \begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$

Solution. Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$, we get

$$\Delta = \begin{vmatrix} 46 & 21 & 219 \\ 42 & 27 & 198 \\ 38 & 17 & 181 \end{vmatrix}$$

Applying $C_1 \to C_1 - 2C_2$ and $C_3 \to C_3 - 10\,C_2$, we get

$$\Delta = \begin{vmatrix} 4 & 21 & 9 \\ -12 & 27 & -72 \\ 4 & 17 & 11 \end{vmatrix}$$

Applying $R_1 \to R_1 - R_3$ and $R_2 \to R_2 + 3R_3$

$$\Delta = \begin{vmatrix} 0 & 4 & -2 \\ 0 & 78 & -39 \\ 4 & 17 & 11 \end{vmatrix} = 2(39) \begin{vmatrix} 0 & 2 & -1 \\ 0 & 2 & -1 \\ 4 & 17 & 11 \end{vmatrix}$$ [Taking 2 common from R_1 and 39 common from R_2]

$= 78 \times 0 = 0$ (Since R_1 and R_2 are identical) **Ans.**

Example 11. *Show that* $\Delta = \begin{vmatrix} b-c & c-a & a-b \\ c-a & a-b & b-c \\ a-b & b-c & c-a \end{vmatrix} = 0$

Solution. Let $$\Delta = \begin{vmatrix} b-c & c-a & a-b \\ c-a & a-b & b-c \\ a-b & b-c & c-a \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 0 & c-a & a-b \\ 0 & a-b & b-c \\ 0 & b-c & c-a \end{vmatrix} = 0$$

Example 12. *Without expanding, evaluate the determinant* $\begin{vmatrix} \sin\alpha & \cos\alpha & \sin(\alpha+\delta) \\ \sin\beta & \cos\beta & \sin(\beta+\delta) \\ \sin\gamma & \cos\gamma & \sin(\gamma+\delta) \end{vmatrix}$.

Solution. Let $$\Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & \sin(\alpha+\delta) \\ \sin\beta & \cos\beta & \sin(\beta+\delta) \\ \sin\gamma & \cos\gamma & \sin(\gamma+\delta) \end{vmatrix}$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & \sin\alpha\cos\delta + \cos\alpha\sin\delta \\ \sin\beta & \cos\beta & \sin\beta\cos\delta + \cos\beta\sin\delta \\ \sin\gamma & \cos\gamma & \sin\gamma\cos\delta + \cos\gamma\sin\delta \end{vmatrix}$$

[Since $\sin(A+B) = \sin A\cos B + \cos A\sin B$]

$\Rightarrow \quad \Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & 0 \\ \sin\beta & \cos\beta & 0 \\ \sin\gamma & \cos\gamma & 0 \end{vmatrix}$ [Applying $C_3 \to C_3 - \cos\delta.C_1 - \sin\delta.C_2$]

$\Rightarrow \quad \Delta = 0$ [Since C_3 consists of all zeros] **Ans.**

Example 13. *Solve the determinantal equation* $\begin{vmatrix} 2x-1 & x+7 & x+4 \\ x & 6 & 2 \\ x-1 & x+1 & 3 \end{vmatrix} = 0$

Solution. Given equation $\begin{vmatrix} 2x-1 & x+7 & x+4 \\ x & 6 & 2 \\ x-1 & x+1 & 3 \end{vmatrix} = 0$

By applying $R_1 \to R_1 - (R_2 + R_3)$, we get $\begin{vmatrix} 0 & 0 & x-1 \\ x & 6 & 2 \\ x-1 & x+1 & 3 \end{vmatrix} = 0$

On expanding by first row, we get

$(x-1)(x^2 + x - 6x + 6) = 0 \Rightarrow (x-1)(x-2)(x-3) = 0 \Rightarrow x = 1, 2, 3$ **Ans.**

Example 14. *Using the properties of determinants, show that*

$$\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = x^3.$$

Solution. Let $\Delta = \begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix}$

Operate : $R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - 3R_1$

$\Delta = \begin{vmatrix} x+y & x & x \\ 3x+2y & 2x & 0 \\ 7x+5y & 5x & 0 \end{vmatrix}$ Expand by C_3 $\Delta = x\begin{vmatrix} 3x+2y & 2x \\ 7x+5y & 5x \end{vmatrix}$

$= x\,[5x(3x+2y) - 2x(7x+5y)]$

$= x\,[15x^2 + 10xy - (14x^2 + 10xy)] = x^3.$ **Proved.**

Example 15. *Using the properties of determinants, evaluate the following :*

$$\begin{vmatrix} 0 & ab^2 & ac^2 \\ a^2b & 0 & bc^2 \\ a^2c & cb^2 & 0 \end{vmatrix}$$

Solution. Let $\Delta = \begin{vmatrix} 0 & ab^2 & ac^2 \\ a^2b & 0 & bc^2 \\ a^2c & cb^2 & 0 \end{vmatrix}$

Take a^2, b^2 and c^2 common from C_1, C_2 and C_3 respectively, $\Delta = a^2b^2c^2\begin{vmatrix} 0 & a & a \\ b & 0 & b \\ c & c & 0 \end{vmatrix}$

Operate : $C_2 \to C_2 - C_3$, $\quad \Delta = a^2 b^2 c^2 \begin{vmatrix} 0 & 0 & a \\ b & -b & b \\ c & c & 0 \end{vmatrix}$

$$\Delta = a^2 b^2 c^2 . a \begin{vmatrix} b & -b \\ c & c \end{vmatrix} = a^3 b^2 c^2 (bc + bc) = 2a^3 b^3 c^3.$$

Example 16. *Using properties of determinants, prove that*

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = x\,y\,z(x - y)(y - z)(z - x).$$

Solution. Let $\quad \Delta = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$

Operate : $C_1 \to C_1 - C_2$; $C_2 \to C_2 - C_3$, $\quad \Delta = xyz \begin{vmatrix} 0 & 0 & 1 \\ x-y & y-z & z \\ x^2 - y^2 & y^2 - z^2 & z^2 \end{vmatrix}$

On expanding by R_1, $\quad \Delta = xyz \begin{vmatrix} x - y & y - z \\ x^2 - y^2 & y^2 - z^2 \end{vmatrix} = xyz(x - y)(y - z) \begin{vmatrix} 1 & 1 \\ x + y & y + z \end{vmatrix}$

$= xyz\,(x - y)\,(y - z)\,(z - x)$. **Proved.**

Example 17. *Using the properties of determinants, show that*

$$\begin{vmatrix} a + x & y & z \\ x & a + y & z \\ x & y & a + z \end{vmatrix} = a^2 (a + x + y + z).$$

Solution. Let $\quad \Delta = \begin{vmatrix} a + x & y & z \\ x & a + y & z \\ x & y & a + z \end{vmatrix}$

Operate : $R_1 \to R_1 - R_2$, $\quad \Delta = \begin{vmatrix} a & -a & 0 \\ x & a + y & z \\ x & y & a + z \end{vmatrix}$

Operate : $C_2 \to C_2 + C_1$, $\quad \Delta = \begin{vmatrix} a & 0 & 0 \\ x & a + y + x & z \\ x & y + x & a + z \end{vmatrix}$

On expanding by R_1 $\quad \Delta = a \begin{vmatrix} a + y + x & z \\ y + x & a + z \end{vmatrix} = a\,[(a + y + x)\,(a + z) - (y + x)\,z]$

$= a\,[a^2 + az + (y + x)\,a + (y + x)\,z - (y + x)\,z]$

$= a^2\,(a + x + y + z)$. **Proved.**

Example 18. *If ω is the one of the imaginary cube roots of unity, find the value of the determinant*

$$\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$$

Solution. The given determinant $= \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$

By $R_1 \to R_1 + R_2 + R_3$, we get

$$= \begin{vmatrix} 1+\omega+\omega^2 & 1+\omega+\omega^2 & 1+\omega+\omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} \quad [\text{Since } 1+\omega+\omega^2 = 0]$$

$= 0$ (Since each entry in R_1 is zero) **Ans.**

Example 19. *Without expanding the determinant, show that $(a + b + c)$ is a factor of* $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$.

Solution. Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Operate : $C_1 \to C_1 + C_2 + C_3$, $\Delta = \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$

⇒ $(a + b + c)$ is a factor of Δ. **Proved.**

Example 20. *Using properties of determinants, prove that* :

$$\begin{vmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix} = 16(3x+4).$$

Solution. Let $\Delta = \begin{vmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$

Operate : $C_1 \to C_1 + C_2 + C_3$, $\Delta = \begin{vmatrix} 3x+4 & x & x \\ 3x+4 & x+4 & x \\ 3x+4 & x & x+4 \end{vmatrix}$

$= (3x+4)\begin{vmatrix} 1 & x & x \\ 1 & x+4 & x \\ 1 & x & x+4 \end{vmatrix} = (3x+4)\begin{vmatrix} 1 & x & x \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{vmatrix}\begin{matrix} \\ R_2 - R_1 \\ R_3 - R_1 \end{matrix} = (3x+4)\begin{vmatrix} 4 & 0 \\ 0 & 4 \end{vmatrix}$

$= 16(3x+4)$ **Proved.**

Example 21. *Without expanding the determinant, prove that* $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0.$

Solution. Let
$$\Delta = \begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$$

Operate : $C_3 \to C_3 + C_2$,
$$\Delta = \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix}$$

$= 0$ ($\because$ C_1 and C_3 are identical). **Proved.**

Example 22. *Evaluate* $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

Solution. By $R_1 \to R_1 + R_2 + R_3$, we get $\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix} \begin{matrix} \\ C_2 - C_1 \\ C_3 - C_1 \end{matrix}$$

On expanding by first row $= (a+b+c)\,(a+b+c)^2 = (a+b+c)^3$. **Ans.**

Example 23. *Show, without expanding* $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x-y)(y-z)(z-x).$

Solution. By $C_1 - C_2$, $C_2 - C_3$, we get $= \begin{vmatrix} 0 & 0 & 1 \\ x-y & y-z & z \\ x^2-y^2 & y^2-z^2 & z^2 \end{vmatrix} = \begin{vmatrix} x-y & y-z \\ x^2-y^2 & y^2-z^2 \end{vmatrix}$

On expanding by first row, we get

$= (x-y)(y-z)\begin{vmatrix} 1 & 1 \\ x+y & y+z \end{vmatrix} = (x-y)\,(y-z)\,(y+z-x-y) = (x-y)\,(y-z)\,(z-x)$. **Proved.**

Example 24. *Prove that* $\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix} = (\alpha-\beta)\,(\beta-\gamma)\,(\gamma-\alpha)\,(\alpha+\beta+\gamma)$

Solution. Let
$$\Delta = \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix}.$$

$$= \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \alpha+\beta+\gamma & \alpha+\beta+\gamma & \alpha+\beta+\gamma \end{vmatrix} \text{[Applying } R_3 \to R_1 + R_3]$$

$$= (\alpha+\beta+\gamma) \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ 1 & 1 & 1 \end{vmatrix} \text{[Taking out } (\alpha+\beta+\gamma) \text{ common from } R_3]$$

$$= (\alpha+\beta+\gamma) \begin{vmatrix} \alpha & \beta-\alpha & \gamma-\alpha \\ \alpha^2 & \beta^2-\alpha^2 & \gamma^2-\alpha^2 \\ 1 & 0 & 0 \end{vmatrix} \text{Applying } C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1$$

$$= (\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha) \begin{vmatrix} \alpha & 1 & 1 \\ \alpha^2 & \beta+\alpha & \gamma+\alpha \\ 1 & 0 & 0 \end{vmatrix}$$

$$= (\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha).1 \begin{vmatrix} 1 & 1 \\ \beta+\alpha & \gamma+\alpha \end{vmatrix} \text{[Expanding along } R_3]$$

$$= (\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha)(\gamma+\alpha-\beta-\alpha)$$

$$= (\alpha+\beta+\gamma)(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)$$ **Proved.**

Example 25. *Show that* $\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3(a+b+c)(ab+bc+ca)$

Solution. Let
$$\Delta = \begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, we get
$$\Delta = \begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & -b+c \\ a+b+c & -c+b & 3c \end{vmatrix}$$

$$=(a+b+c) \begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & -b+c \\ 1 & -c+b & 3c \end{vmatrix} \text{[Taking } (a+b+c) \text{ common from } C_1]$$

$$= (a+b+c) \begin{vmatrix} 1 & -a+b & -a+c \\ 0 & 2b+a & -b+a \\ 0 & -c+a & 2c+a \end{vmatrix} \text{[Applying } R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1]$$

$$= (a+b+c) \begin{vmatrix} 2b+a & -b+a \\ -c+a & 2c+a \end{vmatrix} \text{[Expanding along } C_1]$$

$= (a + b + c)\,[(2b + a)(2c + a) - (-b + a)(-c + a)]$

$= (a + b + c)\,\{(4bc + 2ab + 2ca + a^2 - (bc - ab - ac + a^2)\}$

$= (a + b + c)(3bc + 3ab + 3ca) = 3(a + b + c)(ab + bc + ca)$ **Proved.**

Property (*vi*) If each element of a row (or column) of a determinant consists of the algebraic sum of *n* terms, the determinant can be expressed as the sum of *n* determinants,

$$\text{Let } \Delta = \begin{vmatrix} a_1 + p_1 + q_1 & b_1 & c_1 \\ a_2 + p_2 + q_2 & b_2 & c_2 \\ a_3 + p_3 + q_3 & b_3 & c_3 \end{vmatrix}$$

$= (a_1 + p_1 + q_1)(b_2c_3 - b_3c_2) - (a_2 + p_2 + q_2)(b_1c_3 - b_3c_1) + (a_3 + p_3 + q_3)(b_1c_2 - b_2c_1)$

$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$

$+ p_1(b_2c_3 - b_3c_2) - p_2(b_1c_3 - b_3c_1) + p_3(b_1c_2 - b_2c_1)$

$+ q_1(b_2c_3 - b_3c_2) - q_2(b_1c_3 - b_3c_1) + q_3(b_1c_2 - b_2c_1)$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} p_1 & b_1 & c_1 \\ p_2 & b_2 & c_2 \\ p_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} q_1 & b_1 & c_1 \\ q_2 & b_2 & c_2 \\ q_3 & b_3 & c_3 \end{vmatrix}$$ **Proved.**

Example 31. *If* $\begin{vmatrix} a & a^2 & a^3 - 1 \\ b & b^2 & b^3 - 1 \\ c & c^2 & c^3 - 1 \end{vmatrix} = 0$, *prove that* $abc = 1$.

Solution. $$\begin{vmatrix} a & a^2 & a^3 - 1 \\ b & b^2 & b^3 - 1 \\ c & c^2 & c^3 - 1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} + \begin{vmatrix} a & a^2 & -1 \\ b & b^2 & -1 \\ c & c^2 & -1 \end{vmatrix} = 0$$

$$\Rightarrow \quad abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = 0$$

(Taking out common a, b, c from R_1, R_2 and R_3 from 1st determinant)

$$\Rightarrow \quad abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + \begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix} = 0$$ (Interchanging C_2 and C_3)

$$\Rightarrow \quad abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$$ (Interchanging C_1 and C_2)

$$\Rightarrow \quad (abc - 1)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$$

$\Rightarrow$ abc – 1=0

$\Rightarrow$ abc = 1.

Example 32. *Show that x = – (a + b + c) is one root of the equation*:

$$\begin{vmatrix} x+a & b & c \\ b & x+c & a \\ c & a & x+b \end{vmatrix} = 0$$ *and solve the equation completely.*

Solution. By $C_1 \to C_1 + C_2 + C_3$, we get

$$\Rightarrow \quad (x+a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & x+c & a \\ 1 & a & x+b \end{vmatrix} = 0$$

$$\Rightarrow \quad (x+a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & x-b+c & a-c \\ 0 & a-b & x+b-c \end{vmatrix} = 0, \quad R_2 \to R_2 - R_1;\ R_3 \to R_3 - R_1$$

On expanding by first column, we get

$$(x + a + b + c)\ [(x - b + c)\ (x + b - c) - (a - b)\ (a - c)] = 0$$

$$\Rightarrow \quad (x + a + b + c)\ [x^2 - (b - c)^2 - (a^2 - ac - ab + bc)] = 0$$

$$\Rightarrow \quad (x + a + b + c)\ (x^2 - b^2 - c^2 + 2bc - a^2 + ac + ab - bc] = 0$$

$$\Rightarrow \quad (x + a + b + c)\ (x^2 - a^2 - b^2 - c^2 + ab + bc + ca) = 0$$

Either $x + a + b + c = 0 \Rightarrow x = -(a + b + c)$

or $x^2 - a^2 - b^2 - c^2 + ab + bc + ca = 0$

$$\Rightarrow \quad x = \pm\sqrt{a^2 + b^2 + c^2 - ab - bc - ca}$$

Hence, $x = -(a + b + c)$ is one root of the given equation. **Proved.**

Example 33. *Find the value of* $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$.

Solution. By $C_1 - C_3$, $C_2 - C_3$, we get $\begin{vmatrix} (b+c)^2 - a^2 & a^2 - a^2 & a^2 \\ b^2 - b^2 & (c+a)^2 - b^2 & b^2 \\ c^2 - (a+b)^2 & c^2 - (a+b)^2 & (a+b)^2 \end{vmatrix}$

$$= \begin{vmatrix} (a+b+c)(b+c-a) & 0 & a^2 \\ 0 & (a+b+c)(c+a-b) & b^2 \\ (a+b+c)(c-a-b) & (a+b+c)(c-a-b) & (a+b)^2 \end{vmatrix}$$

On taking out $(a + b + c)$ as common from 1st and 2nd column, we get

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix}$$

$$= (a+b+c)^2 \begin{vmatrix} -a+b+c & 0 & a^2 \\ 0 & a-b+c & b^2 \\ -2b & 2a & 2ab \end{vmatrix} \quad R_3 \to R_3 - (R_1 + R_2)$$

$$= -2(a+b+c)^2 \begin{vmatrix} -a+b+c & 0 & a^2 \\ 0 & a-b+c & b^2 \\ b & a & -ab \end{vmatrix}$$

On expanding by first row, we get

$= -2\,(a + b + c)^2\,[(-a + b + c)\,\{-ab\,(a - b + c) - ab^2\} + a^2\,\{0 - b\,(a - b + c)\}]$

$= -2\,(a + b + c)^2\,[(-a + b + c)\,(-a^2b - abc) - a^2b\,(a - b + c)]$

$= -2ab\,(a + b + c)^2\,[(-a + b + c)\,(-a - c) - a\,(a - b + c)]$

$= -2ab\,(a + b + c)^2\,(a^2 + ac - ab - bc - ac - c^2 - a^2 + ab - ac]$

$= -2ab\,(a + b + c)^2\,(-bc - ac - c^2)$

$= 2abc\,(a + b + c)^2\,(b + a + c)$

$= 2abc\,(a + b + c)^3$. **Ans.**

Example 34. *Using properties of determinants, solve for x :* $\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$

Solution. Given that $\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$

Applying $C_1 \to C_1 + C_2 + C_3$ $\begin{vmatrix} 3a-x & a-x & a-x \\ 3a-x & a+x & a-x \\ 3a-x & a-x & a+x \end{vmatrix} = 0$

$$\Rightarrow (3a-x)\begin{vmatrix} 1 & a-x & a-x \\ 1 & a+x & a-x \\ 1 & a-x & a+x \end{vmatrix} = 0$$

Now, $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, $\Rightarrow (3a-x)\begin{vmatrix} 1 & a-x & a-x \\ 0 & 2x & 0 \\ 0 & 0 & 2x \end{vmatrix} = 0$

Expanding by C_1, we get $(3a - x)\,(4x^2 - 0) = 0$

$\Rightarrow 4x^2\,(3a - x) = 0$ $\Rightarrow$ If $4x^2 = 0$, then $x = 0$

$\Rightarrow$ If $3a - x = 0$, then $x = 3a$

Hence, $x = 0$ or $3a$ **Ans.**

Exercise 2.4

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Prove the following:

1. $\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix} = 0$

2. $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix} = 0.$

3. $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3.$

4. $\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \beta\gamma & \gamma\alpha & \alpha\beta \end{vmatrix} = (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta).$

5. $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

6. $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$

Expand the following determinants, using properties of the determinants:

7. $\begin{vmatrix} 1 & 3 & 7 \\ 4 & 9 & 1 \\ 2 & 7 & 6 \end{vmatrix}$ **Ans.** 51

8. $\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix}$ **Ans.** $(x+2a)(x-a)^2$.

9. Solve the equation

$$\begin{vmatrix} x^3-a^3 & x^2 & x \\ b^3-a^3 & b^2 & b \\ c^3-a^3 & c^2 & c \end{vmatrix} = 0, b \neq c, bc \neq 0.$$

Ans. $x = \frac{a^3}{bc}, x = b, x = c.$

10. Using properties of determinant prove that:

$$\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2.$$

11. Without expanding the determinant, prove that

$$\begin{vmatrix} \frac{1}{a} & a & bc \\ \frac{1}{b} & b & ca \\ \frac{1}{c} & c & ab \end{vmatrix} = 0.$$

12. Without expanding the determinant, prove that

$$\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

2.8 Factor theorem

If the elements of a determinant are polynomials in a variable x and if the substitution $x = a$ makes two rows (or columns) identical, then $(x - a)$ is a factor of the determinant.

When two rows are identical, the value of the determinant is zero. The expansion of a determinant being polynomial in x vanishes on putting $x = a$, then $x - a$ is its factor by the Remainder theorem.

Example 35. *Show that* $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

Solution. If we put $x = y$, $y = z$, $z = x$ then in each case two columns become identical and the determinant vanishes.

$\therefore$ $(x - y), (y - z), (z - x)$ are the factors.

Since the determinant is of third degree, the other factor can be numerical only k (say).

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = k(x-y)(y-z)(z-x) \qquad \text{... (1)}$$

This leading term (product of the elements of the diagonal elements) in the given determinant is yz^2 and in the expansion

$$k(x-y)(y-z)(z-x) \text{ we get } kyz^2$$

Equating the coefficient of yz^2 on both sides of (1), we have

$$k = 1$$

Hence the expansion = $(x - y)(y - z)(z - x)$. **Proved.**

Example 36. *Factorize* $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$

Solution. Putting $a = b$, $C_1 = C_2$ and hence $\Delta = 0$.

$\therefore$ $a - b$ is a factor of Δ. Similarly $b - c$, $c - a$ are also factors of Δ.

$\therefore$ $(a - b)(b - c)(c - a)$ is a third degree factor of Δ which itself is of the fifth degree as is judged from the leading term b^2c^3.

$\therefore$ The remaining factor must be of the second degree. As Δ is symmetrical in a, b, c the remaining factor must, therefore, be of the form $k(a^2 + b^2 + c^2) + l(ab + bc + ca)$

$\therefore$ $\Delta = (a-b)(b-c)(c-a)\{k(a^2+b^2+c^2) + l(ab+bc+ca)\}$

If $k \neq 0$, we shall get terms like a^4b, b^4c etc. which do not occur in Δ. Hence, k must be zero.

$\therefore$ $\Delta = (a-b)(b-c)(c-a)\{0 + l(ab+bc+ca)\}$

or $\Delta = l(a-b)(b-c)(c-a)(ab+bc+ca)$

The leading term in $\Delta = b^2c^3$. The corresponding term on R.H.S = $l\,b^2c^3$

$\therefore$ $l = 1$

Hence, $\Delta = (a-b)(b-c)(c-a)(ab+bc+ca)$. **Ans**

2.9 Conjugate elements

Two equidistant elements lying on a line perpendicular to the leading diagonal are said to be conjugate.

In the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, a_2, b_1; a_3, c_1; b_3, c_2; are pairs of conjugate elements.

2.10 Special types of determinants

2.10.1 Orthosymmetric Determinant. If every element of the leading diagonal is the same and the conjugate elements are equal, then the determinant is said to be orthosymmetric determinant.

$$\begin{vmatrix} a & h & g \\ h & a & f \\ g & f & a \end{vmatrix}$$

2.10.2 Skew-Symmetric Determinant. If the elements of the leading diagonal are all zero and every other element is equal to its conjugate with sign changed, the determinant is said to be Skew- symmetric.

$$\begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

Property 1. A Skew-symmetric determinant of odd order vanishes.

Property 2. A skew-symmetric determinant of even order is a perfect square.

2.11 Application of determinants

Area of triangle. We know that the area of a triangle, whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\,[x_1(y_2 - y_3) - x_2(y_1 - y_3) + x_3(y_1 - y_2)]$$

Note. Since area is always a positive quantity, therefore we always take the absolute value of the determinant for the area.

Condition of collinearity of three points. Let $A\ (x_1, y_1)$, $B\ (x_2, y_2)$ and $C\ (x_3, y_3)$ be three points. Then,

A, B, C are collinear $\Leftrightarrow$ area of triangle $ABC = 0$

$$\Leftrightarrow \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \quad \Leftrightarrow \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Proved.

Example 37. *Using determinants, find the area of the triangle with vertices (– 3, 5), (3, – 6) and (7, 2).*

Solution. The area of the given triangle is

$$\Delta = \frac{1}{2}\begin{vmatrix} -3 & 5 & 1 \\ 3 & -6 & 1 \\ 7 & 2 & 1 \end{vmatrix} = \frac{1}{2}(-3(-6-2) - 5(3-7) + 1(6+42))$$

$$= \frac{1}{2}(24 + 20 + 48) = 46\, unit^2.$$

Example 38. *Using determinants, show that the points (11, 7), (5, 5) and (– 1, 3) are collinear.*

Solution. The area of the triangle formed by the given points is

$$\Delta = \frac{1}{2}\begin{vmatrix} 11 & 7 & 1 \\ 5 & 5 & 1 \\ -1 & 3 & 1 \end{vmatrix} = \frac{1}{2}\big(11(5-3) - 7(5+1) + 1(15+5)\big)$$

$$= \frac{1}{2}(22 - 42 + 20) = 0.$$

It follows that the given three points are lying on a straight line, that means, they are collinear.

Exercise 2.5

Using determinants, find the area of the triangle with vertices:

1. (2,– 7), (1, 3), (10, 8). **Ans.** Area =95

2. (– 1, – 3), (2, 4) and (3, – 1). **Ans.** Area = 11

3. **3.** (1, – 1), (2, 4) and (– 3, 5). **Ans.** Area =13

4. Using determinants, show that the points (3, 8), (– 4, 2) and (10, 14) are collinear.

5. Find the value of α, so that the points (1, – 5), (– 4, 5) and (α, 7) are collinear.

Ans. $\alpha = -5$

6. Find the value of x, if the area of Δ is 35 square cms with vertices (x, 4), (2, – 6), (5, 4).

Ans. $x = -2, 12$

7. Using determinants find the value of k, so that the points (k, 2 – 2k), (– k + 1, 2k) and (– 4 – k, 6 – 2 k) may be collinear.

Ans. k = $-1, \frac{1}{2}$

8. If the points (x, – 2), (5, 2) and (8, 8) are collinear, find x using determinants. **Ans.** $x = 3$

9. If the points (3, – 2), (x, 2) and (8, 8) are collinear, find x using determinants. **Ans.** $x = 1$

2.12 Rule for multiplication of two determinants

Multiply the elements of the first row of Δ_1 with the corresponding elements of the first, the second and the third row of Δ_2 respectively.

Their respective sums form the elements of the first row of $\Delta_1\Delta_2$. Similarly multiply the elements of the second row of Δ_1 with the corresponding elements of first, second and third row of Δ_2 to form the second row of $\Delta_1\Delta_2$ and so on.

Example 39. *Find the product* $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$

Solution. Product of the given determinants
=

$$\begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix} \textbf{ Ans.}$$

Example 40. *Prove that the determinant*

$$\begin{vmatrix} 2b_1 + c_1 & c_1 + 3a_1 & 2a_1 + 3b_1 \\ 2b_2 + c_2 & c_2 + 3a_2 & 2a_2 + 3b_2 \\ 2b_3 + c_3 & c_3 + 3a_3 & 2a_3 + 3b_3 \end{vmatrix}$$

is a multiple of the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \textit{and find the other factor.}$$

Solution.

$$\begin{vmatrix} 2b_1 + c_1 & c_1 + 3a_1 & 2a_1 + 3b_1 \\ 2b_2 + c_2 & c_2 + 3a_2 & 2a_2 + 3b_2 \\ 2b_3 + c_3 & c_3 + 3a_3 & 2a_3 + 3b_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} 0 & 2 & 1 \\ 3 & 0 & 1 \\ 2 & 3 & 0 \end{vmatrix}$$

Ans.

Example 41. *Prove that*

$$\begin{vmatrix} 1 & \cos(\beta - \alpha) & \cos(\gamma - \alpha) \\ \cos(\alpha - \beta) & 1 & \cos(\gamma - \beta) \\ \cos(\alpha - \gamma) & \cos(\beta - \gamma) & 1 \end{vmatrix} = 0$$

Solution.

$$\begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix} \times \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix} = 0$$

or

$$\begin{vmatrix} \cos^2\alpha + \sin^2\alpha & \cos\alpha\cos\beta + \sin\alpha\sin\beta & \cos\alpha\cos\gamma + \sin\alpha\sin\gamma \\ \cos\beta\cos\alpha + \sin\beta\sin\alpha & \cos^2\beta + \sin^2\beta & \cos\beta\cos\gamma + \sin\beta\sin\gamma \\ \cos\gamma\cos\alpha + \sin\gamma\sin\alpha & \cos\gamma\cos\beta + \sin\gamma\sin\beta & \cos^2\gamma + \sin^2\gamma \end{vmatrix} = 0$$

or

$$\begin{vmatrix} 1 & \cos(\beta - \alpha) & \cos(\gamma - \alpha) \\ \cos(\alpha - \beta) & 1 & \cos(\gamma - \beta) \\ \cos(\alpha - \gamma) & \cos(\beta - \gamma) & 1 \end{vmatrix} = 0$$

Proved

Miscellaneous MCQ Exercises

1. The determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by:

(A) a + d
(B) ad - bc
(C) ab + cd
(D) ac - bd
Answer: (B) ad - bc

2. If the determinant of a square matrix is zero, then the matrix is:

(A) Singular
(B) Non-singular
(C) Invertible
(D) Identity matrix
Answer: (A) Singular

3. If $A = \begin{bmatrix} 2 & 3 \\ 4 & x \end{bmatrix}$ and det(A) = 10, then the value of x is:

(A) 1
(B) 2
(C) 3
(D) 11
Answer: (D) 11

4. If det(A) = 5, then the determinant of kA for a 3×3 matrix is:

(A) $5k^3$
(B) $5k^2$
(C) $5k^4$
(D) $5k$
Answer: (A) $5k^3$

5. If A is a 3×3 matrix such that det(A) = 7, then what is det(A^{-1})?

(A) 7
(B) 1/7
(C) 0
(D) -7
Answer: (B) 1/7

6. If $A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$ then det(A) is:
(A) 0
(B) 1
(C) 2
(D) 3
Answer: (B) 3

7. If det(A) = 3 and det(B) = 4 for two 2×2 matrices A and B, then det(AB) is:

(A) 7
(B) 12
(C) 1
(D) 0
Answer: (B) 12

8. The determinant of an identity matrix of any order is:

(A) 0
(B) 1
(C) -1
(D) Depends on the order
Answer: (B) 1

9. If a row or column of a determinant is multiplied by a scalar k, then the determinant is:

(A) Unchanged
(B) Multiplied by k
(C) Multiplied by k^n (where n is the order of the matrix)
(D) Multiplied by k^{n-1}
Answer: (C) Multiplied by k^n

10. The determinant of a triangular matrix (upper or lower) is:

(A) The sum of diagonal elements
(B) The product of diagonal elements
(C) Always 0
(D) Always 1
Answer: (B) The product of diagonal elements

Chapter 3

Matrices (II)

3.1 Minors, Cofactors, Determinants and Adjoint of a matrix

Minors associated with elements of a square matrix

A minor of each element of a square matrix is the unique value of the determinant associated with it, which is obtained after eliminating the row and column in which the element exists.

For a 2×2 matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$M_{11} = a_{22}$, $M_{12} = a_{21}$, $M_{21} = a_{12}$, $M_{22} = a_{11}$

For a 3×3 matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$,…, $M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

Cofactors associated with elements of a square matrix

The cofactor of each element is obtained on multiplying its minor by $(-1)^{i+j}$.

$C_{ij} = (-1)^{i+j} M_{ij}$

Determinant of a square matrix

Every square matrix is associated with a determinant and is denoted by det (A) or $|A|$.

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

Determinant of order n can be expanded by any one row or column using the formula $|A| = \sum_{j=1}^{n} a_{ij}\, C_{ij}$, where C_{ij} is the cofactor corresponding to the element a_{ij} .

A determinant of order 2 is evaluated as:

$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

A determinant of order 3 is evaluated as:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \sum_{j=1}^{n}(-1)^{i+j}a_{ij}M_{ij}$$

$$=a_{11}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} - a_{12}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + a_{13}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$=a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

Note: A determinant may be evaluated using any row or column, value remains the same.

Properties of Determinants

- Value of a determinant remains unchanged if rows and columns are interchanged i.e. $|A| = |A^T|$
- If any two rows or columns are interchanged, the value of determinant is multiplied by (-1)
- The value of determinant remains unchanged if k times elements of a row (column) is added to another row (column).
- If elements in any row (column) in a determinant are multiplied by a scalar k, then value of determinant is multiplied by k. Thus, if each element in the determinant is multiplied by k, value of determinant of order n multiplies by k^n i.e., $|kA| = k^n|A|$
- If A and B are square matrices of same order, then $|AB| = |A||B|$

Adjoint of a square matrix

The adjoint of a square matrix A of order n is the transpose of the matrix of cofactors of each element. If C_{11} , C_{12} , C_{13} ,..., C_{nn} be the cofactors of elements a_{11} , a_{12} , a_{13} ,..., a_{nn} of the matrix A. Then adjoint of A is given by

$$adj(A) = \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \dots & \dots & \dots & \dots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{pmatrix}^T = \begin{pmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \dots & \dots & \dots & \dots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{pmatrix}$$

3.2 Inverse of a Matrix

The inverse of a square matrix A of order n, denoted by A^{-1} is such that

$AA^{-1} = A^{-1}A = I_n$ where I_n is an identity matrix of order n.

A matrix is invertible if and only if matrix is non-singular i.e., $|A| \neq 0$. There are many methods to find inverse of a square matrix.

3.2.1 Inverse of a matrix using adjoint

Working rule to find inverse of a matrix using adjoint:

1. Calculate $|A|$
   i. If $|A| = 0$, inverse does not exist

ii. if $|A| \neq 0$, go to step 2

2. Find $adj(A)$ and compute the inverse using the formula $A^{-1} = \frac{adj(A)}{|A|}$

Example1 Find inverse of the matrix $\begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$

Solution: Let $A = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$

$$|A| = 1(16-9) - 3(4-3) + 3(3-4) = 1(7) - 3(1) + 3(-1) = 1$$

$$adj(A) = \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \dots & \dots & \dots & \dots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{pmatrix}^T = \begin{pmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \dots & \dots & \dots & \dots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{pmatrix}$$

$C_{11} = (-1)^2(16-9) = 7$ $\quad C_{12} = (-1)^3(4-3) = -1$ $\quad C_{13} = (-1)^4(3-4) = 1$

$C_{21} = (-1)^3(12-9) = -3$ $\quad C_{22} = (-1)^4(4-3) = 1$ $\quad C_{23} = (-1)^5(3-3) = 0$

$C_{31} = (-1)^4(9-12) = -3$ $\quad C_{32} = (-1)^5(3-3) = 0$ $\quad C_{33} = (-1)^6(4-3) = 1$

$$\therefore adj(A) = \begin{pmatrix} 7 & -1 & 1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}^T = \begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$A^{-1} = \frac{adj(A)}{|A|} = \frac{1}{1}\begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

3.2.2 Inverse of a matrix by using Gauss-Jordan method

To find the inverse of a matrix using Gauss-Jordan method, we take an augmented matrix $(A : I)$ and transform it into another augmented matrix $(I : A)$ using elementary row (column) transformations.

Elementary row (column) transformations: As the name suggests, row (columns) operations are executed on matrices according to certain set of rules such that the transformed matrix is equivalent to the original matrix. These rules are:

- Any two rows (columns) are interchangeable i.e., $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$
- All the elements of any row (column) can be multiplied by any non-zero number k i.e., $R_i \rightarrow kR_i$
- All the elements of a row (column) can be added one to one to corresponding scalar multiples of another row (column) i.e., $R_i \rightarrow R_i + kR_j$

Working rule to find inverse of a matrix using Gauss-Jordan method:

1. Prepare an augmented matrix $(A : I)$

2. Using elementary row transformations make element (1,1) of the augmented matrix as 1, and using this make all other elements in the 1st column zero.
3. Now make the element (2,2) as 1, using row transformations and remaining elements in the 2nd column zero.
4. Continue the process until the augmented matrix is transformed to $(I : A^{-1})$

Note: (i) Do not apply row and column transformations to the same matrix while using Gauss-Jordan method.

(ii) While using column transformations make element (1,1) of the augmented matrix as 1, and using this make all other elements in the 1st row zero and similarly proceed for other rows.

(iii) In the process of forming identity matrix, ensure that previously formed zeros and ones are not altered while applying row (column) transformations. For this while making element (2,2) as 1, do not use R_1 (C_1) and while making element (3,3) as one neither use R_1 (C_1) nor R_2 (C_2).

Example 2 Find the inverse of the following matrices using Gauss-Jordan method

(i) $\begin{pmatrix} 3 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 3 \end{pmatrix}$ (ii) $\begin{pmatrix} 3 & -1 & 2 \\ 2 & -1 & 3 \\ -1 & 1 & 1 \end{pmatrix}$

Solution: (i) Let $A = \begin{pmatrix} 3 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 3 \end{pmatrix}$

Augmented matrix is: $\begin{pmatrix} 3 & 1 & 3 & : & 1 & 0 & 0 \\ 3 & 1 & 4 & : & 0 & 1 & 0 \\ 4 & 1 & 3 & : & 0 & 0 & 1 \end{pmatrix}$

Transforming element at (1,1) position to one

$R_1 \to -R_1 + R_2$ $\begin{pmatrix} 1 & 0 & 0 & : & -1 & 0 & 1 \\ 3 & 1 & 4 & : & 0 & 1 & 0 \\ 4 & 1 & 3 & : & 0 & 0 & 1 \end{pmatrix}$

Making element at (2,1) and (3,1) positions as 0

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 4R_1$ $\begin{pmatrix} 1 & 0 & 0 & : & -1 & 0 & 1 \\ 0 & 1 & 4 & : & 3 & 1 & -3 \\ 0 & 1 & 3 & : & 4 & 0 & -3 \end{pmatrix}$

Elements at (2,2) and (1,2) are already 1 and 0, so making element at (3,2) position to zero

$R_3 \to R_3 - R_2$ $\begin{pmatrix} 1 & 0 & 0 & : & -1 & 0 & 1 \\ 0 & 1 & 4 & : & 3 & 1 & -3 \\ 0 & 0 & -1 & : & 1 & -1 & 0 \end{pmatrix}$

Now transforming element at (3,3) position to one

$R_3 \to -R_3$ $\qquad \begin{pmatrix} 1 & 0 & 0 & : & -1 & 0 & 1 \\ 0 & 1 & 4 & : & 3 & 1 & -3 \\ 0 & 0 & 1 & : & -1 & 1 & 0 \end{pmatrix}$

Element at (1,3) is 0, so Transforming element at (2,3) position to zero

$$R_2 \to R_2 - 4R_3 \quad \begin{pmatrix} 1 & 0 & 0 & : & -1 & 0 & 1 \\ 0 & 1 & 0 & : & 7 & -3 & -3 \\ 0 & 0 & 1 & : & -1 & 1 & 0 \end{pmatrix} = (I : A^{-1})$$

$$\therefore \ A^{-1} = \begin{pmatrix} -1 & 0 & 1 \\ 7 & -3 & -3 \\ -1 & 1 & 0 \end{pmatrix}$$

(ii) Let $A = \begin{pmatrix} 3 & -1 & 2 \\ 2 & -1 & 3 \\ -1 & 1 & 1 \end{pmatrix}$

Augmented matrix is: $\begin{pmatrix} 3 & -1 & 2 & : & 1 & 0 & 0 \\ 2 & -1 & 3 & : & 0 & 1 & 0 \\ -1 & 1 & 1 & : & 0 & 0 & 1 \end{pmatrix}$

Transforming element at (1,1) position as 1

$R_1 \to R_1 - R_2$ $\qquad \begin{pmatrix} 1 & 0 & -1 & : & 1 & -1 & 0 \\ 2 & -1 & 3 & : & 0 & 1 & 0 \\ -1 & 1 & 1 & : & 0 & 0 & 1 \end{pmatrix}$

Transforming element at (2,1) and (3,1) positions as 0

$R_2 \to R_2 - 2R_1 \, , R_3 \to R_3 + R_1$ $\quad \begin{pmatrix} 1 & 0 & -1 & : & 1 & -1 & 0 \\ 0 & -1 & 5 & : & -2 & 3 & 0 \\ 0 & 1 & 0 & : & 1 & -1 & 1 \end{pmatrix}$

Transforming element at (2,2) position as 1

$R_2 \to -R_2$ $\qquad \begin{pmatrix} 1 & 0 & -1 & : & 1 & -1 & 0 \\ 0 & 1 & -5 & : & 2 & -3 & 0 \\ 0 & 1 & 0 & : & 1 & -1 & 1 \end{pmatrix}$

Element at (1,2) is 0, so transforming element at (3,2) position to zero

$R_3 \to R_3 - R_2$ $\qquad \begin{pmatrix} 1 & 0 & -1 & : & 1 & -1 & 0 \\ 0 & 1 & -5 & : & 2 & -3 & 0 \\ 0 & 0 & 5 & : & -1 & 2 & 1 \end{pmatrix}$

Now making element at (3,3) position 1

$R_3 \to \frac{1}{5} R_3$ $\qquad \begin{pmatrix} 1 & 0 & -1 & : & 1 & -1 & 0 \\ 0 & 1 & -5 & : & 2 & -3 & 0 \\ 0 & 0 & 1 & : & -\frac{1}{5} & \frac{2}{5} & \frac{1}{5} \end{pmatrix}$

Now transforming elements at (1,3) and (2,3) positions 0

$$R_1 \to R_1 + R_3\ ,\ R_2 \to R_2 + 5R_3 \begin{pmatrix} 1 & 0 & 0 & : & \frac{4}{5} & -\frac{3}{5} & \frac{1}{5} \\ 0 & 1 & 0 & : & 1 & -1 & 1 \\ 0 & 0 & 1 & : & -\frac{1}{5} & \frac{2}{5} & \frac{1}{5} \end{pmatrix} = (I : A^{-1})$$

$$\therefore\ A^{-1} = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} & \frac{1}{5} \\ 1 & -1 & 1 \\ -\frac{1}{5} & \frac{2}{5} & \frac{1}{5} \end{pmatrix}$$

3.3 Solution of System of Linear Simultaneous Equations

Here we will be discussing some direct methods of solving a system of linear equations.

3.3.1 Matrix Method

Working rule to solve a system of equations using matrix method

1. Write the system of equations as $AX = B$
2. Calculate $|A|$
   i. If $|A| = 0$, system of equations can not be solved using matrix method
   ii. if $|A| \neq 0$, go to step 3
3. Find $adj(A)$ and compute the inverse using the formula $A^{-1} = \frac{adj(A)}{|A|}$
4. Solution of the system of equations is given by $X = A^{-1}B$

Example3 Solve the system of equations using matrix method

$$x + 3y + 2z = 5$$
$$2x + 4y - 6z = -4$$
$$x + 5y + 3z = 10$$

Solution: Let the system of equations be represented as $AX = B$

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 4 & -6 \\ 1 & 5 & 3 \end{pmatrix},\ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}\ ,\ B = \begin{pmatrix} 5 \\ -4 \\ 10 \end{pmatrix}$$

$$|A| = 1(12 + 30) - 2(9 - 10) + 1(-18 - 8) = 18 \neq 0$$

$C_{11} = (-1)^2(12 + 30) = 42$ $\quad C_{12} = (-1)^3(6 + 6) = -12$ $\quad C_{13} = (-1)^4(10 - 4) = 6$

$C_{21} = (-1)^3(9 - 10) = 1$ $\quad C_{22} = (-1)^4(3 - 2) = 1$ $\quad C_{23} = (-1)^5(5 - 3) = -2$

$C_{31} = (-1)^4(-18 - 8) = -26$ $\quad C_{32} = (-1)^5(-6 - 4) = 10$ $\quad C_{33} = (-1)^6(4 - 6) = -2$

$$\therefore adj(A) = \begin{pmatrix} 42 & -12 & 6 \\ 1 & 1 & -2 \\ -26 & 10 & -2 \end{pmatrix}^T = \begin{pmatrix} 42 & 1 & -26 \\ -12 & 1 & 10 \\ 6 & -2 & -2 \end{pmatrix}$$

$$A^{-1} = \frac{adj(A)}{|A|} = \frac{1}{18}\begin{pmatrix} 42 & 1 & -26 \\ -12 & 1 & 10 \\ 6 & -2 & -2 \end{pmatrix}$$

$$X = A^{-1}B = \frac{1}{18}\begin{pmatrix} 42 & 1 & -26 \\ -12 & 1 & 10 \\ 6 & -2 & -2 \end{pmatrix}\begin{pmatrix} 5 \\ -4 \\ 10 \end{pmatrix} = \frac{1}{18}\begin{pmatrix} 42(5)+1(-4)-26(10) \\ -12(5)+1(-4)+10(10) \\ 6(5)-2(-4)-2(10) \end{pmatrix} = \frac{1}{18}\begin{pmatrix} -54 \\ 36 \\ 18 \end{pmatrix}$$

$\therefore x = -3, y = 2, z = 1$

3.3.2 Gauss Elimination Method

Working rule to solve system of equations using Gauss Elimination method

1. Write the system of equations as $AX = B$
2. Write the matrix in augmented form as $C = (A : B)$
3. Reduce matrix A in $C = (A : B)$ to echelon form using row transformations
4. Solve the system of equations $AX = B$ by backward substitution method.

Example4 Solve the system of equations using Gauss Elimination method

$$x + 3y + 2z = 5$$
$$2x + 4y - 6z = -4$$
$$x + 5y + 3z = 10$$

Solution: Let the system of equations be represented as $AX = B$

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 4 & -6 \\ 1 & 5 & 3 \end{pmatrix},\ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix},\ B = \begin{pmatrix} 5 \\ -4 \\ 10 \end{pmatrix}$$

Augmented matrix $C = \begin{pmatrix} 1 & 3 & 2 & : & 5 \\ 2 & 4 & -6 & : & -4 \\ 1 & 5 & 3 & : & 10 \end{pmatrix}$

Transforming element at (2,1) and (3,1) positions as 0

$$R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - R_1 \quad \begin{pmatrix} 1 & 3 & 2 & : & 5 \\ 0 & -2 & -10 & : & -14 \\ 0 & 2 & 1 & : & 5 \end{pmatrix}$$

Transforming element at (2,2) to one

$$R_2 \to R_2/-2 \quad \begin{pmatrix} 1 & 3 & 2 & : & 5 \\ 0 & 1 & 5 & : & 7 \\ 0 & 2 & 1 & : & 5 \end{pmatrix}$$

Transforming element at (3,2) to zero

$$R_3 \to R_3 - 2R_2 \quad \begin{pmatrix} 1 & 3 & 2 & : & 5 \\ 0 & 1 & 5 & : & 7 \\ 0 & 0 & -9 & : & -9 \end{pmatrix}$$

$\therefore$ Corresponding system of equations is given as

$$x + 3y + 2z = 5 \quad \ldots (1)$$
$$4y + 5z = 7 \quad \ldots (2)$$
$$-9z = -9 \quad \ldots (3)$$

Solving by back substitution

③⇒$z = 1$, using $z = 1$ in ② ⇒$y = 2$, using $y = 2,\ z = 1$ in ①⇒$x = -3$

∴ $x = -3, y = 2, z = 1$ is the required solution of given system of equations

3.3.3 Gauss Jordan Elimination Method

Working rule to solve system of equations using Gauss Jordan Elimination method

1. Write the system of equations as $AX = B$
2. Write the matrix in augmented form as $C = (A\!:B)$
3. Apply elementary row transformations to reduce the matrix A in $C = (A\!:B)$ to unit matrix
4. Last column of the transformed matrix augmented matrix gives vector X.

Example5 Solve the system of equations using Gauss Jorden Elimination method

$$x + 3y + 2z = 5$$
$$2x + 4y - 6z = -4$$
$$x + 5y + 3z = 10$$

Solution: Let the system of equations be represented as $AX = B$

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 4 & -6 \\ 1 & 5 & 3 \end{pmatrix},\ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}\ ,B = \begin{pmatrix} 5 \\ -4 \\ 10 \end{pmatrix}$$

Augmented matrix $C = \begin{pmatrix} 1 & 3 & 2 & : & 5 \\ 2 & 4 & -6 & : & -4 \\ 1 & 5 & 3 & : & 10 \end{pmatrix}$

Transforming element at (2,1) and (3,1) positions as 0

$$R_2 \to R_2 - 2R_1\ ,R_3 \to R_3 - R_1 \quad \begin{pmatrix} 1 & 3 & 2 & : & 5 \\ 0 & -2 & -10 & : & -14 \\ 0 & 2 & 1 & : & 5 \end{pmatrix}$$

Transforming element at (2,2) to one

$$R_2 \to R_2/-2 \quad \begin{pmatrix} 1 & 3 & 2 & : & 5 \\ 0 & 1 & 5 & : & 7 \\ 0 & 2 & 1 & : & 5 \end{pmatrix}$$

Transforming elements at (1,2) and (3,2) to zero

$$R_1 \to R_1 - 3R_2, R_3 \to R_3 - 2R_2 \quad \begin{pmatrix} 1 & 0 & -13 & : & -16 \\ 0 & 1 & 5 & : & 7 \\ 0 & 0 & -9 & : & -9 \end{pmatrix}$$

Transforming element at (3,3) to one

$$R_3 \to R_3/-9 \quad \begin{pmatrix} 1 & 0 & -13 & : & -16 \\ 0 & 1 & 5 & : & 7 \\ 0 & 0 & 1 & : & 1 \end{pmatrix}$$

Transforming elements at (1,3) and (2,3) to zero

$$R_1 \to R_1 + 13R_3, R_2 \to R_2 - 5R_3 \quad \begin{pmatrix} 1 & 0 & 0 & : & -3 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & 1 \end{pmatrix}$$

$\therefore x = -3, y = 2, z = 1$ is the required solution of given system of equations

3.4 Rank of a Matrix

The rank of a matrix A is the order of the highest ordered non-zero minor in A. It is denoted by $\rho(A)$.

Example6 Find the rank of the following matrices:

(i) $A = \begin{pmatrix} 1 & 4 \\ 3 & 7 \end{pmatrix}$ (ii) $A = \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}$ (iii) $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{pmatrix}$

Solution: (i) Here $\begin{vmatrix} 1 & 4 \\ 3 & 7 \end{vmatrix} = -5 \neq 0 \ \therefore \ \rho(A) = 2$

(ii) Here $\begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix} = 0 \qquad \therefore \ \rho(A) = 1$

(iii) Here $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{vmatrix} = 0 \quad \therefore \ \rho(A) \neq 3$

Next consider $\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = 2 \neq 0 \ \therefore \ \rho(A) = 2$

3.4.1 Rank of a matrix using Normal form

The normal form of a matrix is one of the following:

I_r, $(I_n, 0)$, $\begin{pmatrix} I_n \\ 0 \end{pmatrix}$, $(I_n \quad 0)$ or $\begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}$ where I_n is the identity matrix of order n.

Changing to normal form, n is the rank of the given matrix.

Example7 Find the rank of the following matrices by reducing them to normal form:

(i) $A = \begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$ (ii) $A = \begin{pmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 \\ 1 & 5 & 0 & 1 \end{pmatrix}$

Solution: (i) Transforming element at (1,1) position to unity

Applying $R_1 \to \frac{1}{3} R_1$, we get $A \sim \begin{pmatrix} 1 & -1 & 4/3 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$

Transforming element at (2,1) position to zero

$R_2 \to R_2 - 2R_1$ $\qquad A \sim \begin{pmatrix} 1 & -1 & 4/3 \\ 0 & -1 & 4/3 \\ 0 & -1 & 1 \end{pmatrix}$

Transforming element at (1,2) and (1,3) position to zero

$C_2 \to C_2 + C_1$, $C_3 \to C_3 - \frac{4}{3}C_1$ $\quad A \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 4/3 \\ 0 & -1 & 1 \end{pmatrix}$

Transforming element at (2,2) position to one

$R_2 \to -R_2$ $\qquad A \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -4/3 \\ 0 & -1 & 1 \end{pmatrix}$

Transforming element at (3,2) position to zero

$R_3 \to R_3 + R_2$ $\quad A \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -4/3 \\ 0 & 0 & -1/3 \end{pmatrix}$

Transforming element at (2,3) position 0

$C_3 \to C_3 + \frac{4}{3}C_2$ $\quad A \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/3 \end{pmatrix}$

Transforming element at (3,3) position 1

$C_3 \to -3C_3$ $\quad A \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\qquad$ Hence rank of the given matrix is 3.

(ii) Here $A = \begin{pmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 \\ 1 & 5 & 0 & 1 \end{pmatrix}$

Transforming element at (2,1) and (2,2) position to zero

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ we get $A \sim \begin{pmatrix} 1 & 3 & 4 & 5 \\ 0 & -1 & 2 & 2 \\ 0 & 2 & -4 & -4 \end{pmatrix}$

Transforming element at (1,2) , (1,3) and (1,4) position 0

$C_2 \to C_2 - 3C_1$, $C_3 \to C_3 - 4C_1$, $C_4 \to C_4 - 5C_1$ $\quad A \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 2 \\ 0 & 2 & -4 & -4 \end{pmatrix}$

Making element at (2,2) position 1

$R_2 \to -R_2 \qquad A \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & -2 \\ 0 & 2 & -4 & -4 \end{pmatrix}$

Making element at (3,2) position 0

$R_3 \to R_3 - 2R_2 \quad A \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Making element at (2,3) and (2,4) position 0

$C_3 \to C_3 + 2C_2\ ,\ C_4 \to C_4 + 2C_2 \quad A \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Hence rank of the given matrix is 2.

3.4.2 Rank of a matrix using Echelon form

Echelon Form: A matrix is said to be in Echelon form if:

(i) The number of zeros in succeeding row are greater than previous row

(ii) The first non-zero entry in each non-zero row is equal to unity.

Working rule: Transform the matrix to echelon form. The number of non-zero rows in echelon form becomes the rank of the matrix.

Example8 Find the rank of the following matrices by reducing them to echelon form:

(i) $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{pmatrix}$ (ii) $A = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 2 & 3 & 4 & 5 \end{pmatrix}$

Solution: (i) Transforming elements at (2,1) and (3,1) positions to zeros

Applying $R_2 \to R_2 - 2R_1\ ,\ R_3 \to R_3 - 3R_1$ we get

$A \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Making element at (3,3) position 0

$R_3 \to R_3 - R_2 \ A \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Now the matrix is reduced to echelon form. Since the number of non-zero rows is 2, hence the rank of the given matrix is 2.

(ii) Transforming elements at (2,1) and (3,1) positions to zeros

Applying $R_2 \to R_2 - 2R_1\ ,\ R_3 \to R_3 - 2R_1$, we get

$A \sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & -1 & -2 & 1 \end{pmatrix}$

Making element at (2,2) position 1

$R_2 \to -R_2 \qquad A \sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & -1 & -2 & 1 \end{pmatrix}$

Making element at (3,2) position 0

$R_3 \to R_3 + R_2 \quad A \sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & 4 \end{pmatrix}$

Making element at (3,3) position 1

$R_3 \to -R_3 \qquad A \sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & -4 \end{pmatrix}$

Now the matrix is reduced to echelon form. Since the number of non-zero rows is 3, hence the rank of the given matrix is 3.

3.4.3 Linear Dependence and Independence of Vectors

The set of vectors $X_{1,}X_{2,}X_{3,} \dots, X_n$ is said to be linearly dependent if there exist scalars $C_{1,}C_{2,}C_{3,} \dots, C_n$ not all zero , such that $C_1X_1 + C_2X_2 + C_3X_3 + \cdots + C_nX_n = 0$ And they are linearly independent if $C_1X_1 + C_2X_2 + C_3X_3 + \cdots + C_nX_n = 0$

$\Rightarrow C_i = 0 \ \forall i = 1,2,3,\dots,n$

Example9 Examine the following system of vectors for linear dependence. If dependent find the relation between them:

(i) $X_1 = [1 \quad -1 \quad 1]$, $X_2 = [2 \quad 1 \quad 1]$ and $X_3 = [3 \quad 0 \quad 2]$

(ii) $X_1 = [1 \quad 2 \quad 3]$ and $X_2 = [2 \quad -2 \quad 6]$

Solution: (i) Consider $C_1X_1 + C_2X_2 + C_3X_3 = 0$……….(1)

$\Rightarrow C_1[1 \quad -1 \quad 1] + C_2[2 \quad 1 \quad 1] + C_3[3 \quad 0 \quad 2] = 0$

$\Rightarrow C_1 + 2C_2 + 3C_3 = 0$

$-C_1 + C_2 + 0C_3 = 0$

$C_1 + C_2 + 2C_3 = 0$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Applying $R_2 \to R_2 + R_1$, $R_3 \to R_3 - R_1$, we get

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Applying $R_3 \to R_3 + \frac{1}{3}R_2$, we get

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow\ C_1 + 2C_2 + 3C_3 = 0$

$0C_1 + 3C_2 + 3C_3 = 0$

Let $C_3 = k$ $\Rightarrow\ C_2 = -k$ and $C_1 = -k$

Hence the given vectors are linearly dependent.

Putting these values in (1) , we get $-kX_1 - kX_2 + kX_3 = 0$

$\Rightarrow -k(X_1 + X_2 - X_3) = 0$

$\Rightarrow X_1 + X_2 - X_3 = 0$

which is the required relation between them.

(i) Consider $C_1X_1 + C_2X_2 = 0$ ....(1)

$\Rightarrow C_1[1 \quad 2 \quad 3] + C_2[2 \quad -2 \quad 6]$

$\Rightarrow C_1 + 2C_2 = 0$

$2C_1 - 2C_2 = 0$

$3C_1 + 6C_2 = 0$

$\Rightarrow\ C_1 = 0$ and $C_2 = 0$

Hence the given vectors are linearly independent.

3.5 **Consistency and Inconsistency of Linear System of Equations**

Consider $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$

......

$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$

This is the system of *m* equations in *n* unknowns and it can be written in the form $AX = B$ where

$$A = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots & a_{1n} \\ a_{21} & a_{22} \ldots\ldots & a_{2n} \\ a_{31} & a_{32} \ldots\ldots & a_{3n} \\ & .. & \\ & .. & \\ a_{m1} & a_{m2} \ldots\ldots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ .. \\ .. \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ .. \\ .. \\ b_m \end{bmatrix}$$

Here if $b_i = 0\ \forall i$ then system of equations is said to be homogeneous otherwise it is non-homogeneous.

The matrix $C = [A:B] = \begin{bmatrix} a_{11} & a_{12} \dots\dots & a_{1n} : b_1 \\ a_{21} & a_{22} \dots\dots & a_{2n} : b_2 \\ a_{31} & a_{32} \dots\dots & a_{3n} : b_3 \\ & \vdots & \\ a_{m1} & a_{m2} \dots\dots & a_{mn} : b_m \end{bmatrix}$ is called augmented matrix.

3.5.1 Working Rule to find solution of Non-Homogeneous System of Equations

1. For the system of equations, $AX = B$, form an augmented matrix $C = [A:B]$.
2. Find the ranks of matrix A and matrix C

i. If Rank of $A \neq$ Rank of C, then the given system is inconsistent i.e., it has no solution.

ii. If Rank of $A =$ Rank of $C =$ Number of variables in the given system of equations, then the system has a unique solution.

iii. If Rank of $A =$ Rank of $C <$ Number of variables in the given system of equations, then the system has infinitely many solutions.

3.5.2 Working Rule to find solution of Homogeneous System of Equations i.e. *AX = 0*

In case of homogeneous equations, $b_i = 0\ \forall i$**,** therefore augmented matrix is not required. Here we find the ranks of matrix A

i. If Rank of $A =$ Number of variables in the given system of equations, then the system has the trivial solution, i.e., $x_1 = x_2 = \cdots x_n = 0$

ii. If Rank of A is less than the number of variables in the given system of equations, then the system has infinitely many solutions.

Example Show that the following system of equations is inconsistent.

$$x + 2y + z = 2$$
$$3x + y - 2z = 1$$
$$4x - 3y - z = 3$$
$$2x + 4y + 2z = 5$$

Solution: Let the system of equations be represented as $AX = B$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & -2 \\ 4 & -3 & -1 \\ 2 & 4 & 2 \end{pmatrix},\ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix},\ B = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 5 \end{pmatrix}$$

Transforming elements at (2,1) , (3,1) and (4,1) positions to zeros

Applying $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 4R_1$, $R_4 \to R_4 - 2R_1$, we get

$$A \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & -5 & -5 \\ 0 & -11 & -5 \\ 0 & 0 & 0 \end{pmatrix}$$

Now matrix A has a non-zero minor $\begin{vmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & -11 & -5 \end{vmatrix} = -30$

$\therefore$ Rank of matrix A is 3

Again $C = [A\colon B] = \begin{pmatrix} 1 & 2 & 1 & : & 2 \\ 3 & 1 & -2 & : & 1 \\ 4 & -3 & -1 & : & 3 \\ 2 & 4 & 2 & : & 5 \end{pmatrix}$

Applying $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 4R_1$, $R_4 \to R_4 - 2R_1$, we get

$$C = [A\colon B] \sim \begin{pmatrix} 1 & 2 & 1 & : & 2 \\ 0 & -5 & -5 & : & -5 \\ 0 & -11 & -5 & : & -5 \\ 0 & 0 & 0 & : & 1 \end{pmatrix}$$

$$[C] = 1 \begin{vmatrix} 2 & 1 & 2 \\ -5 & -5 & -5 \\ -11 & -5 & -5 \end{vmatrix} + 0 = 2(0) + 5(-5 + 10) - 11(-5 - 10) = 190$$

$\therefore$ Rank of matrix $C = [A\colon B]$ is 4

Hence the given system of equations is inconsistent.

Exercise

MCQ

1. The rank of a matrix is:

   (A) The number of nonzero rows in its row echelon form
   (B) The number of zero rows in its row echelon form
   (C) The number of columns in the matrix
   (D) The determinant of the matrix

   Answer: (A)

2. A square matrix is invertible if and only if:

   (A) Its determinant is zero
   (B) Its rank is equal to its order
   (C) It has at least one zero row
   (D) It is a diagonal matrix

   Answer: (B)

3. The inverse of a matrix A exists if:

   (A) $\text{rank}(A) = 0$
   (B) $\det(A) = 0$
   (C) $\text{rank}(A) = n$, where A is an $n \times n$ matrix
   (D) None of the above

   Answer: (C)

4. The rank of a 3×3 identity matrix is:

   (A) 0
   (B) 1
   (C) 2
   (D) 3

   Answer: (D)

5. If A is a singular matrix, then:

   (A) A is invertible
   (B) $\det(A) = 0$
   (C) A is diagonalizable
   (D) A has full rank

   Answer: (B)

6. The rank of an $m \times n$ matrix is at most:

   (A) $m + n$
   (B) $\min(m, n)$
   (C) $\max(m, n)$
   (D) $m \times n$

   Answer: (B)

7. The inverse of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by:

   (A) $\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
   (B) $\frac{1}{ad+bc} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$
   (C) $\frac{1}{ad-bc} \begin{bmatrix} a & -c \\ -b & d \end{bmatrix}$
   (D) $\frac{1}{ad+bc} \begin{bmatrix} d & b \\ c & a \end{bmatrix}$

   Answer: (A)

8. If a matrix has two identical rows, then its determinant is:

   (A) Nonzero
   (B) Zero
   (C) One
   (D) None of the above

   Answer: (B)

9. The inverse of a matrix A satisfies:

   (A) $AA^{-1} = 0$
   (B) $AA^{-1} = A$
   (C) $AA^{-1} = I$

(D) $A^{-1}A = 0$

Answer: (C)

10. The rank of a zero matrix is:

 (A) 0
 (B) 1
 (C) Equal to the number of columns
 (D) Equal to the number of rows

 Answer: (A)

11. If A is an $n \times n$ invertible matrix, then its rank is:

 (A) 0
 (B) n
 (C) $n - 1$
 (D) None of the above

 Answer: (B)

12. A matrix is invertible if:

 (A) It is singular
 (B) It has linearly dependent rows
 (C) It has full rank
 (D) It has at least one zero row

 Answer: (C)

13. If the determinant of a square matrix is nonzero, then:

 (A) The matrix is singular
 (B) The matrix is invertible
 (C) The rank of the matrix is less than its order
 (D) None of the above

 Answer: (B)

14. The rank of an upper triangular matrix is:

 (A) The number of nonzero diagonal elements
 (B) The number of zero diagonal elements
 (C) Always equal to the order of the matrix
 (D) None of the above

Answer: (A)

15. If A is an invertible matrix, then $(A^{-1})^{-1}$ is:

    (A) A

    (B) A^T

    (C) A^{-2}

    (D) None of the above

    Answer: (A)

Short answer questions

1. Given a square matrix A, what condition must A satisfy to have an inverse?

2. Compute the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

3. If A and B are invertible matrices of the same order, is AB always invertible? Justify your answer.

4. If the rank of an $m \times n$ matrix A is equal to the smaller of m or n, what can you conclude about A?

5. Find the inverse of the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, if it exists.

Long answer questions

1. **Inverse of a Matrix:** Given a square matrix A, define what it means for A to be invertible. Derive the formula for the inverse of a 2×2 matrix and provide an example to illustrate the process of finding the inverse.

2. **Rank of a Matrix:** Define the rank of a matrix. Explain the concept of row rank and column rank and prove that they are always equal. Also, discuss the importance of rank in determining the solution of a system of linear equations.

3. **Finding the Inverse Using Elementary Row Operations:** Describe the procedure for finding the inverse of a matrix using elementary row operations. Apply this method to compute the inverse of the matrix

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

   if it exists.

4. **Rank and Linear Dependence:** Explain the relationship between the rank of a matrix and the linear dependence of its rows and columns. Given the matrix

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

determine its rank and justify your answer.

5. **Applications of Matrix Inverse:** Discuss the applications of the inverse of a matrix in solving systems of linear equations. Use the inverse matrix method to solve the system:

$$\begin{aligned} 2x + 3y &= 5 \\ 4x + y &= 6 \end{aligned}$$

Chapter 4

Systems of Linear Equations

4.1 Introduction:

Systems of linear equations play an important and motivating role in the subject of linear algebra. In fact, many problems in linear algebra reduce to finding the solution of a system of linear equations. Thus, the techniques introduced in this chapter will be applicable to abstract ideas introduced later. On the other hand, some of the abstract results will give us new insights into the structure and properties of systems of linear equations.

4.2 Systems of Linear Equations in Two Variables:

Linear systems are a fundamental part of linear algebra, a subject used in most modern mathematics. Computational algorithms for finding the solutions are an important part of numerical linear algebra, and play a prominent role in engineering, physics, chemistry, computer science, and economics. A system of non-linear equations can often be approximated by a linear system, a helpful technique when making a mathematical model or computer simulation of a relatively complex system.

To establish basic concepts of Linear Systems, let's consider the following simple example: If 2 adult tickets and 1 child ticket cost 32, and if 1 adult ticket and 3 child tickets cost 36, what is the price of each?

How to find it ?

Let: x = price of adult ticket

y = price of child ticket

Then:

$$2x + y = 32$$
$$x + 3y = 36$$

Now we have a system of two linear equations in two variables. It is easy to find ordered pairs (x, y) that satisfy one or the other of these equations. For example, the ordered pair (16, 0) satisfies the first equation but not the second, and the ordered pair (24, 4) satisfies the second but not the first.

To solve this system, we must find all ordered pairs of real numbers that satisfy both equations at the same time. In general, we have the following definition:

Systems of Linear Equations in Two Variables: Systems of Two Linear Equations in Two Variables Given the linear system

$$ax + by = h$$
$$cx + dy = k$$

where a, b, c, d, h, and k are real constants, a pair of numbers $x = x_0$ and $y = y_0$ also written as an ordered pair $(x_0 \ , y_0)$ is a solution of this system if each equation is satisfied by the pair. The set of all such ordered pairs is called the **solution set** for the system. To solve a system is to find its solution set.

A system of linear equations is **consistent** if it has one or more solutions and **inconsistent** if no solutions exist. Furthermore, a consistent system is said to be **independent,** if it has exactly one solution (often referred to as the **unique solution**) and **dependent,** if it has more than one solution. Two systems of equations are equivalent if they have the same solution set.

Possible Solutions to a Linear System: The linear system

$$ax + by = h$$
$$cx + dy = k$$

must have (i) Exactly one solution (Consistent and independent)

or

(ii) No solution (Inconsistent)

or

(iii) Infinitely many solutions (Consistent and dependent)

There are no other possibilities.

We will consider three methods of solving such systems:

1. Graphing,
2. Substitution,
3. Elimination by addition.

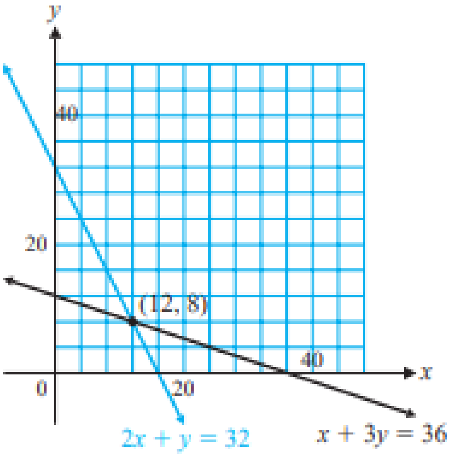
Each method has its advantages, depending on the situation.

Graphing: Recall that the graph of a line is a graph of all the ordered pairs that satisfy the equation of the line. To solve the ticket problem by graphing, we graph both equations in the same coordinate system. The coordinates of any points that the graphs have in common must be solutions to the system since they satisfy both equations.

►Example: (Solving a System by Graphing) Solve the ticket problem by graphing:

$$2x + y = 32$$
$$x + 3y = 36$$

Solution: An easy way to find two distinct points on the first line is to find the x and y intercepts. Substitute $y = 0$ to find the x-intercept, $2x = 32$, so $x = 16$, and substitute $x = 0$ to find the y-intercept y = 32. Then draw the line through (16, 0) and (0, 32). After graphing both lines in the same coordinate system (Fig. 1), estimate the coordinates of the intersection point:



(Fig. 1)

$x = 12$, Price of Adult ticket

$y = 8$, Price of Child ticket

►Exercise: Solve each of the following systems by graphing:

(1) $x - 2y = 2$, $x + y = 5$ (2) $x + 2y = -4$, $2x + 4y = 8$ (3) $2x + 4y = 8$, $x + 2y = 4$

Substitution: Now we review an algebraic method that is easy to use and provides exact solutions to a system of two equations in two variables, provided that solutions exist. In this method, first we choose one of two equations in a system and solve for one variable in terms of the other. (We make a choice that avoids fractions, if possible.) Then we substitute the result into the other equation and solve the resulting linear equation in one variable. Finally, we substitute this result back into the results of the first step to find the second variable.

►Example: (Solving a System by Substitution) Solve by substitution:

$$5x + y = 4$$
$$2x - 3y = 5$$

Solution: Solve either equation for one variable in terms of the other; then substitute into the remaining equation. In this problem, we avoid fractions by choosing the first equation and solving for y in terms of x as-

$5x + y = 4$ (Solve the first equation for y in terms of x.)

$\rightarrow \quad y = 4 - 5x$ (Substitute into the second equation.)

Now, $2x - 3y = 5$ (Second equation)

$\rightarrow 2x - 3(4 - 5x) = 5$ (Solve for x, i.e. putting the value of y)

$\rightarrow \quad 2x - 12 + 15x = 5$

$\rightarrow \quad 17x = 17$

$\rightarrow \quad x = 1$

Now, replace x with 1 in $y = 4 - 5x$ to find y:

$y = 4 - 5x$

$\rightarrow \quad y = 4 - 5(1)$ (Replace x with 1)

$\rightarrow \quad y = -1$

Hence the solution is $x = 1,\ y = -1$ or (1, -1)

►Exercise: Solve by substitution:

$$3x + 2y = -2$$
$$2x - y = -6$$

Elimination by Addition: The methods of graphing and substitution both work well for systems involving two variables. However, neither is easily extended to larger systems. Now we turn to elimination by addition. This is probably the most important method of solution. It readily generalizes to larger systems and forms the basis for computer-based solution methods.

To solve an equation such as $2x - 5 = 3$, we perform operations on the equation until we reach an equivalent equation whose solution is obvious.

$2x - 5 = 3$ (Add 5 to both sides)

$\rightarrow \quad 2x = 8$ (Divide both sides by 2.)

$\rightarrow \quad x = 4$

Operations That Produce Equivalent Systems: A system of linear equations is transformed into an equivalent system if

(1) Two equations are interchanged.
(2) An equation is multiplied by a nonzero constant.

(3) A constant multiple of one equation is added to another equation

Any one of the above three operations in can be used to produce an equivalent system, but the operations that will be of most use to us now we discus. The use of above rule is best illustrated by examples.

►Example: (Solving a System Using Elimination by Addition) Solve the following system using elimination by addition:

$$3x - 2y = 8$$
$$2x + 5y = -1$$

Solution: We use above rule to eliminate one of the variables, obtaining a system with an obvious solution:

$$3x - 2y = 8 \quad \text{..............(i)}$$
$$2x + 5y = -1 \quad \text{.............(ii)}$$

Multiply the equation (i) by 5 and the equation (ii) by 2 we get,

$$5(3x - 2y) = 5(8)$$
$$2(2x + 5y) = 2(-1)$$

i.e.,

$$15x - 10y = 40$$
$$4x + 10y = -2$$

Add the top equation to the bottom equation and eliminating the y terms we get,

$$19x = 38$$

Divide both sides by 19, $x = 2$

Knowing that $x = 2$, we substitute this number back into either of the two original equations (we choose the second) to solve for y:

$$2\,(2) + 5y = -1$$
$$\rightarrow \quad 5y = -5$$
$$\rightarrow \quad y = -1$$

Hence the solution is $x = 2,\ y = -1$ or $(2, -1)$

►Exercise: Solve the following system using elimination by addition:

$$5x - 2y = 12$$
$$2x + 3y = 1$$

Applications: Many real-world problems are solved readily by constructing a mathematical model consisting of two linear equations in two variables and applying the solution methods that we have discussed. We shall examine two applications in detail.

►Exercise: Jasmine wants to use milk and orange juice to increase the amount of calcium and vitamin A in her daily diet. An ounce of milk contains 37 milligrams of calcium and 57 micrograms of vitamin A. An ounce of orange juice contains 5 milligrams of calcium and 65 micrograms of vitamin A. How many ounces of milk and orange juice should Jasmine drink each day to provide exactly 500 milligrams of calcium and 1,200 micrograms of vitamin A?

Most linear systems of any consequence involve large numbers of equations and variables. It is impractical to try to solve such systems by hand. In the past, these complex systems could be solved only on large computers. Now there are a wide array of approaches to solving linear systems, ranging from graphing calculators to software and spreadsheets. In the rest of this chapter, we develop several matrix methods for solving systems with the understanding that these methods are generally used with a graphing calculator. It is important to keep in mind that we are not presenting these techniques as efficient methods for solving linear systems by hand. Instead, we emphasize formulation of mathematical models and interpretation of the results, two activities that graphing calculators cannot perform for you.

4.3 linear systems:

In mathematics, a **system of linear equations** (or **linear systems**) is a collection of two or more linear equations involving the same variables. For example, linear equations involving the variables x, y, z may be in the form-

$$\begin{aligned} x + y + z &= 6 \\ y + 3z &= 11 \\ x - 2y + z &= 0 \end{aligned}$$

Each of the equations, from the systems of linear equations are called linear equation. So, Systems of linear equations can be considered as a collection of linear equations.

A **linear equation** in variables $x_1,\ x_2,\ x_3, \dots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where $a_1, a_2, \dots, a_n$ and b are constant real complex numbers. The constant a_i is called the **coefficient** of x_i and b is called the **constant term** of the equation.

A **system of linear equations** (or **linear system**) is a finite collection of linear equations in same variables. For instance, a linear system of m equations in n variables x_1, $x_2,\ x_3, \dots, x_n$ can be written as-

$$a_{11}x_1 + a_{12}x_2 + \dots\dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots\dots + a_{2n}x_n = b_2$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots\dots + a_{mn}x_n = b_m$$

[linear system (1.1)]

A **solution** of a linear system (1.1) is a tuple (s_1, s_2, ..., s_n) of numbers that makes each equation a true statement when the values s_1, s_2, ... ,s_n are substituted for $x_2,\ x_3, \dots, x_n$, respectively. The set of all solutions of a linear system is called the **solution set** of the system.

Theorem: Any system of linear equations has one of the following exclusive conclusions.

(1) No solution.

(2) Unique solution.

(3) Infinitely many solutions.

A linear system is said to be **consistent** if it has at least one solution and is said to be **inconsistent** if it has no solution.

4.4 Geometric interpretation:

The following three linear systems

$$(a)\dots\begin{cases}(i)\ \ 2x_1 & +x_2 & = 3\\ (ii)\ 2x_1 & -x_2 & = 0\\ (iii)\ \ x_1 & -2x_2 & = 4\end{cases}$$

$$(b)\dots\begin{cases}(i)\ \ 2x_1 & +x_2 & = 3\\ (ii)\ 2x_1 & -x_2 & = 5\\ (iii)\ \ x_1 & -2x_2 & = 4\end{cases}$$

$$(c)\dots\begin{cases}(i)\ \ 2x_1 & +x_2 & = 3\\ (ii)\ 4x_1 & +2x_2 & = 6\\ (iii)\ 6x_1 & +3x_2 & = 9\end{cases}$$

Have no solution, a unique solution, and infinitely many solutions, respectively. See the below Figure.

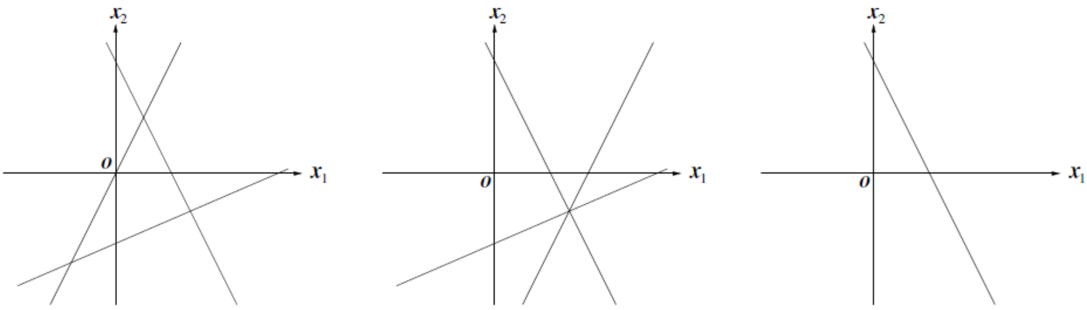


Figure 1: No solution, unique solution, and infinitely many solutions.

Note: A linear equation of two variables represents a straight line in R^2. A linear equation of three variables represents a plane in R^3. In general, a linear equation of *n* variables represents a hyper-plane in the *n*-dimensional Euclidean space R^n.

4.5 Solution of linear equations by determinants (Cramer's rule):

In linear algebra, Cramer's rule is an explicit formula for the solution of a system of linear equations with as many equations as unknowns, valid whenever the system has a unique solution. It expresses the solution in terms of the determinants of the (square) coefficient matrix and of matrices obtained from it by replacing one column by the column vector of right-sides of the equations. It is named after Gabriel Cramer, who published the rule for an arbitrary number of unknowns in 1750, although Colin Maclaurin also published special cases of the rule in 1748 and possibly knew of it as early as 1729.

Cramer's rule, implemented in a naive way, is computationally inefficient for systems of more than two or three equations

Let us solve the following equations.

$$a_1 x + b_1 y + c_1 z = d_1$$

$$a_2 x + b_2 y + c_2 z = d_2$$

$$a_3 x + b_3 y + c_3 z = d_3$$

Let us write these equations in the form $AX = B$.

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Let,

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } D_1 = \begin{vmatrix} D_1 & b_1 & c_1 \\ D_2 & b_2 & c_2 \\ D_3 & b_3 & c_3 \end{vmatrix}, \ D_2 = \begin{vmatrix} a_1 & D_1 & c_1 \\ a_2 & D_2 & c_2 \\ a_3 & D_3 & c_3 \end{vmatrix}, \ D_3 = \begin{vmatrix} a_1 & b_1 & D_1 \\ a_2 & b_2 & D_2 \\ a_3 & b_3 & D_3 \end{vmatrix}$$

Then, $x = \frac{D_1}{D}$, , $y = \frac{D_2}{D}$, , $z = \frac{D_3}{D}$

►Example: Solve the following system of equations using Cramer's rule:

$$5x - 7y + z = 11$$

$$6x - 8y - z = 15$$

$$3x + 2y - 6z = 7$$

Solution: The given systems of linear equations are,

$$5x - 7y + z = 11$$

$$6x - 8y - z = 15$$

$$3x + 2y - 6z = 7$$

Here, $D = \begin{vmatrix} 5 & -7 & 1 \\ 6 & -8 & -1 \\ 3 & 2 & -6 \end{vmatrix} = 5\,(48 + 2) + 7\,(-36 + 3) + 1\,(12 + 24) = 55\ (\neq 0)$

$$D_1 = \begin{vmatrix} 11 & -7 & 1 \\ 15 & -8 & -1 \\ 7 & 2 & -6 \end{vmatrix} = 11\,(48 + 2) + 7\,(-90 + 7) + 1\,(30 + 56) = 55$$

$$D_2 = \begin{vmatrix} 5 & 11 & 1 \\ 6 & 15 & -1 \\ 3 & 7 & -6 \end{vmatrix} = 5\,(-90 + 7) - 11\,(-36 + 3) + 1\,(42 - 45) = -55$$

$$D_3 = \begin{vmatrix} 5 & -7 & 11 \\ 6 & -8 & 15 \\ 3 & 2 & 7 \end{vmatrix} = 5\,(-56 - 30) + 7\,(42 - 45) + 11\,(12 + 24) = -55$$

Hence, $x = \frac{D_1}{D} = \frac{55}{55} = 1,$, $y = \frac{D_2}{D} = \frac{-55}{55} = -1,$, $z = \frac{D_3}{D} = \frac{-55}{55} = 1$

►Example: Solve the following system of equations using Cramer's rule:

$$x + y + z = 6$$
$$y + 3z = 11$$
$$x - 2y + z = 0$$

Solution: The given systems of linear equations are,

$$x + y + z = 6$$
$$y + 3z = 11$$
$$x - 2y + z = 0$$

Let us write these equations in the form $AX = B$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$$

Now, D = $\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{vmatrix} = 1(1+6) - 1(0-3) + 1(0-1)$ = 7+3-1= 9 $\neq$ 0

As, D $\neq$ 0. So the given system of equations has a unique solution.

Also,

$$D_1 = \begin{vmatrix} 6 & 1 & 1 \\ 11 & 1 & 3 \\ 0 & -2 & 1 \end{vmatrix} = 6(1+6) - 1(11-0) + 1(-22-0) = 42 - 11 - 22 = 9$$

$$D_2 = \begin{vmatrix} 1 & 6 & 1 \\ 0 & 11 & 3 \\ 1 & 0 & 1 \end{vmatrix} = 1(11-0) - 6(0-3) + 1(0-11) = 11 + 18 - 11 = 18$$

$$D_3 = \begin{vmatrix} 1 & 1 & 6 \\ 0 & 1 & 11 \\ 1 & -2 & 0 \end{vmatrix} = 1(0+22) - 1(0-11) + 6(0-1) = 22 + 11 - 6 = 27$$

Hence, $x = \frac{D_1}{D} = \frac{9}{9} = 1,$, $y = \frac{D_2}{D} = \frac{18}{9} = 2,$, $z = \frac{D_3}{D} = \frac{27}{9} = 3$

►Exercise: Solve, by determinants, the following set of simultaneous equations

$$5x - 6y + 4z = 15$$

$$7x + 4y - 3z = 19$$

$$2x + y + 6z = 46$$

►Exercise: Solve the following system of equations using Cramer's Rule:

$$2x - 3y + 4z = -9$$

$$-3x + 4y + 2 = -12$$

$$4x - 2y - 3z = -3$$

►Exercise: The sum of three numbers is 6. If we multiply the third number by 2 and add the first number to the result, we get 7. By adding second and third numbers to three times the first number we get 12. Use determinants to find the numbers.

4.6 Matrix Inversion Method

This method can be applied only when the coefficient matrix is a square matrix and non-singular. Consider the matrix equation, $AX = B$ ……(i)

Where A is a non-singular square matrix and. Since A is non-singular, A^{-1} exists and have the properties $A^{-1}A = AA^{-1} = I$. Pre-multiplying both sides of (i) by A^{-1}, we get $A^{-1}(AX) = A^{-1}B$. That is, $(A^{-1}A)X = A^{-1}B$. Hence, we get $X = A^{-1}B$.

►Example : Solve the following system of equations, using matrix inversion method:

$$2x_1 + 3x_2 + 3x_2 = 5,$$

$$x_1 - 2x_2 + x_2 = -4,$$

$$3x_1 - x_2 - 2x_2 = 3$$

Solution: The matrix form of the system is $AX = B$, where

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix},\ X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},\ B = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}.$$

Now $\det A = \begin{vmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{vmatrix} = 2(4+1) - 3(-2-3) + 3(-1+6) = 10 + 15 + 15 = 40 \neq 0$

So, A^{-1} exist.

Now, $A^{-1} = \frac{1}{\det A}(adj\ A)$

$$= \frac{1}{40}\begin{bmatrix} +(4+1) & -(-2-3) & +(-1+6) \\ -(-6+3) & +(-4-9) & -(-2-9) \\ +(3+6) & -(2-3) & +(-4-3) \end{bmatrix}^T = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

Then applying $X = A^{-1}B$, we get

$$X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

$$= \frac{1}{40}\begin{bmatrix} 25-12+27 \\ 25+52+3 \\ 25-44-21 \end{bmatrix}$$

$$= \frac{1}{40}\begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

So, the solution is $x_1 = 1$, $x_2 = 2$, $x_3 = -1$.

►Exercise: Solve the following equations by matrix inversion method:

$$x + y + z = 4$$
$$2x - y + 3z = 1$$
$$3x + 2y - z = 1$$

4.7 Matrices of a linear system:

In solving systems of equations using elimination by addition, the coefficients of the variables and the constant terms played a central role. The process can be made more efficient for generalization and computer work by the introduction of a mathematical form called a matrix. A matrix is a rectangular array of numbers written within brackets.

Matrix notation in a spreadsheet: Matrices serve as shorthand for solving systems of linear equations. Associated with the system

$$2x - 3y = 5$$
$$x + 2y = -3$$

are its coefficient matrix, constant matrix, and augmented matrix:

$$\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 5 \\ -3 \end{bmatrix} \qquad \left[\begin{array}{cc|c} 2 & -3 & 5 \\ 1 & 2 & -3 \end{array}\right]$$

Coefficient matrix Constant matrix Augmented matrix

Note that the augmented matrix is just the coefficient matrix, augmented by the constant matrix. The vertical bar is included only as a visual aid to separate the coefficients from the constant terms. The augmented matrix contains all of the essential information about the linear system—everything but the names of the variables.

For ease of generalization to the larger systems in later sections, we will change the notation for the variables in above system to a subscript form. That is, in place of x and y, we use x_1 and x_2, respectively, and system is rewritten as-

$$2x_1 - 3x_2 = 5$$

$$x_1 + 2x_2 = -3$$

In general, associated with each linear system of the form

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2$$

The Augment matrix of the system is: $\begin{bmatrix} a_{11} & a_{12} & | & b_1 \\ a_{21} & a_{22} & | & b_2 \end{bmatrix}$

This matrix contains the essential parts of above system of linear equations. Our objective is to learn how to manipulate augmented matrices in order to solve system, if a solution exists. The manipulative process is closely related to the elimination process discussed in 4.2.

Now we consider more generalized system-

Definition: The **augmented matrix** of the general linear system (1.1) is the table

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & | & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & | & b_2 \\ . & . & \cdots & . & | & . \\ . & . & \cdots & . & | & . \\ a_{m1} & a_{m2} & \cdots & a_{mn} & | & b_m \end{bmatrix}$$

and the **coefficient matrix** of (1.1) is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Recall that two linear systems are said to be equivalent if they have the same solution set. We used the operations listed below to transform linear systems into equivalent systems:

(1) Two equations are interchanged.

(2) An equation is multiplied by a nonzero constant.

(3) A constant multiple of one equation is added to another equation.

Paralleling the earlier discussion, we say that two augmented matrices are row equivalent, denoted by the symbol ~, placed between the two matrices, if they are augmented matrices of equivalent systems of equations. How do we transform augmented matrices into row-equivalent matrices?

4.8 Elementary row operations:

In mathematics, an elementary matrix is a square matrix obtained from the application of a single elementary row operation to the identity matrix. The elementary matrices generate the general linear group $GL_n(F)$ when F is a field. Left multiplication (pre-multiplication) by an elementary matrix represents elementary row operations, while right multiplication (post-multiplication) represents elementary column operations.

Definition: There are three kinds of elementary row operations on matrices:

(a) Adding a multiple of one row to another row;

(b) Multiplying all entries of one row by a non-zero constant;

(c) Inter changing two rows.

There are three types of elementary matrices, which correspond to three types of row operations (respectively, column operations):

Row switching

A row within the matrix can be switched with another row.

Row multiplication

Each element in a row can be multiplied by a non-zero constant. It is also known as *scaling* a row.

Row addition

A row can be replaced by the sum of that row and a multiple of another row.

Definition: Two linear systems in same variables are said to be **equivalent** if their solution sets are the same. A matrix A is said to be **row equivalent** to a matrix B, written $A \sim B$, If there is a sequence of elementary row operations that changes A to B.

Theorem: If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set. In other words, elementary row operations do not change solution set.

4.9 Row echelon forms:

Definition: A matrix is said to be in **row echelon form** if it satisfies the following two conditions:

(i) All zero rows are gathered near the bottom.

(ii) The first non zero entry of a row, called the **leading entry** of that row, is a head of the first non-zero entry of the next row.

A matrix in row echelon form is said to be in **reduced row echelon form** if it satisfies two more conditions:

(i) The leading entry of every non zero row is 1.

(ii) Each leading entry 1 is the only non-zero entry in its column.

A matrix in (reduced) row echelon form is called a **(reduced) row echelon matrix**.

Note: Sometimes we call row echelon forms just as echelon forms and row echelon matrices as echelon matrices without mentioning the word "row."

4.10 Row echelon form pattern:

The following are two typical row echelon matrices.

$$\begin{bmatrix} \bullet & * & * & * & * & * & * & * & * \\ 0 & \bullet & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \bullet & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \bullet & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \bullet & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where the circled $\bullet$ represent arbitrary non-zero numbers, and the stars * represent arbitrary numbers, including zero.

The following are two typical reduced row echelon matrices.

$$\begin{bmatrix} 1 & 0 & * & * & 0 & * & 0 & * & * \\ 0 & 1 & * & * & 0 & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & * & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition: If a matrix A is row equivalent to a row echelon matrix B, we say that A has the **row echelon form** B ; if B is further are deduced row echelon matrix, then we say that A has the **reduced row echelon form** B.

4.11 Row reduction algorithm:

Definition: A **pivot position** of a matrix A is a location of entries of A that corresponds to a leading entry in a row echelon form of A. A **pivot column** (**pivot row**) is a column (row) of A that contains a pivot position.

Algorithm (Row Reduction Algorithm):

(1) Begin with the left most non-zero column, which is a **pivot column**; the top entry is **pivot position**.

(2) If the entry of the pivot position is zero, select a nonzero entry in the pivot column, interchange the pivot row and the row containing this nonzero entry.

(3) If the pivot position is nonzero, use elementary row operations to reduce all entries below the pivot position to zero, (and the pivot position to 1 and entries above the pivot position to zero for reduced row echelon form).

(4) Cover the **pivot row** and the rows above it; repeat (1)-(3) to the remaining sub-matrix.

Theorem: Every matrix is row equivalent to one and only one reduced row echelon matrix. In other words, every matrix has a unique reduced row echelon form.

4.12 Solving linear system:

► **Example:** Find all solutions for the linear system

$$x_1 + 2x_2 - x_3 = 1$$
$$2x_1 + x_2 + 4x_3 = 2$$
$$3x_1 + 3x_2 + 4x_3 = 1$$

Solution: Perform the row operations:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 1 & 4 & 2 \\ 3 & 3 & 4 & 1 \end{array}\right] \begin{array}{c} R_2 - 2R_1 \\ \sim \\ R_3 - 3R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -3 & 6 & 0 \\ 0 & -3 & 7 & -2 \end{array}\right] \begin{array}{c} (-1/3)R_2 \\ \sim \\ R_3 - R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{c} R_1 + R_3 \\ \sim \\ R_2 + 2R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{c} R_1 - 2R_2 \\ \sim \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

The system is equivalent to

$$\begin{cases} x_1 &= 7 \\ x_2 &= -4 \\ x_3 &= -2 \end{cases}$$

which means the system has a unique solution.

△

► **Exercise:** Find all solutions for the linear system

$$\begin{array}{ccccccc} x_1 & -x_2 & +x_3 & -x_4 & = & 2 \\ x_1 & -x_2 & +x_3 & +x_4 & = & 0 \\ 4x_1 & -4x_2 & +4x_3 & & = & 4 \\ -2x_1 & +2x_2 & -2x_3 & +x_4 & = & -3 \end{array}$$

► **Exercise:** Find all solutions for the linear system

$$2x_1 + x_2 - x_3 = 1$$
$$x_1 + 3x_2 + 4x_3 = 2$$
$$7x_1 + 3x_2 + 4x_3 = 1$$

Theorem: A linear system is consistent if and only if the row echelon form of its augmented matrix contains no row of the form

$$[0 \quad ,\dots, \quad 0 \quad | \quad b], \text{ where } b \neq 0$$

►Example: Solve the linear system whose augmented matrix is :

$$A = \begin{bmatrix} 0 & 0 & 1 & -1 & 2 & 1 & | & 0 \\ 3 & 6 & 0 & 3 & -3 & 2 & | & 7 \\ 1 & 2 & 0 & 1 & -1 & 0 & | & 1 \\ 2 & 4 & -2 & 4 & -6 & -5 & | & -4 \end{bmatrix}$$

Solution: Interchanging Row 1 and Row 3, we have

$$\begin{bmatrix} 1 & 2 & 0 & 1 & -1 & 0 & | & 1 \\ 3 & 6 & 0 & 3 & -3 & 2 & | & 7 \\ 0 & 0 & 1 & -1 & 2 & 1 & | & 0 \\ 2 & 4 & -2 & 4 & -6 & -5 & | & -4 \end{bmatrix}$$

$$\underset{R_4' = R_4 - 2R_1}{\overset{R_2' = R_2 - 3R_1}{\sim}}$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & -1 & 0 & | & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & | & 4 \\ 0 & 0 & 1 & -1 & 2 & 1 & | & 0 \\ 0 & 0 & -2 & 2 & -4 & -5 & | & -6 \end{bmatrix}$$

$$\overset{R_2 \leftrightarrow R_3}{\sim}$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & -1 & 0 & | & 1 \\ 0 & 0 & 1 & -1 & 2 & 1 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & | & 4 \\ 0 & 0 & -2 & 2 & -4 & -5 & | & -6 \end{bmatrix}$$

$$\overset{R_4' = R_4 + 2R_2}{\sim}$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & -1 & 0 & | & 1 \\ 0 & 0 & 1 & -1 & 2 & 1 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & | & 4 \\ 0 & 0 & 0 & 0 & 0 & -3 & | & -6 \end{bmatrix}$$

$$\overset{R_3' = \frac{1}{2}R_3}{\sim}$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & -1 & 0 & | & 1 \\ 0 & 0 & 1 & -1 & 2 & 1 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & | & 2 \\ 0 & 0 & 0 & 0 & 0 & -3 & | & -6 \end{bmatrix}$$

$$R_4' = R_4 + 3R_2$$
$$\sim$$
$$\begin{bmatrix} 1 & 2 & 0 & 1 & -1 & 0 & | & 1 \\ 0 & 0 & 1 & -1 & 2 & 1 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & | & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$
$$R_2' = R_2 - R_3$$
$$\sim$$
$$\begin{bmatrix} 1 & 2 & 0 & 1 & -1 & 0 & | & 1 \\ 0 & 0 & 1 & -1 & 2 & 0 & | & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & | & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Then the system is equivalent to,

$$x_1 + 2x_2 + x_4 - x_5 = 1$$
$$x_3 - x_4 + 2x_5 = -2$$
$$x_6 = 2$$

This is same as,

$$x_1 = 1 - 2x_2 - x_4 + x_5$$
$$x_3 = -2 + x_4 - 2x_5$$
$$x_6 = 2$$

The unknowns x_2, x_4 and x_5 are free variables. Set $x_2 = c_1$, $x_4 = c_2$, $x_5 = c_3$, where c_1, c_2, c_3 are arbitrary. The general solutions of the system are given by-

$$\begin{cases} x_1 = 1 - 2c_1 - c_2 + c_3 \\ x_2 = c_1 \\ x_3 = -2 + c_2 - 2c_3 \\ x_4 = c_2 \\ x_5 = c_3 \\ x_6 = 2 \end{cases}$$

So, the system has infinitely many solutions.

Definition: A variable in a consistent linear system is called free if its corresponding column in the Coefficient matrix is not a pivot column.

Theorem: For any homogeneous system $Ax = 0$,
#{variables}= #{pivot positions of A} + #{free variables}

4.13 Applications:

Systems of linear equations are used in various real-life applications across different fields. Here are some examples:

1. Business and Economics

- **Cost and Revenue Analysis:** A company wants to determine the break-even point where total cost equals total revenue.
  Example: $C = 50x + 2000C \ \ and\ R = 80x$

  Solving $C = R$ helps determine the number of units needed to break even.

- **Investment and Budgeting:** If a person invests in two stocks with different rates of return, they can use a system of equations to allocate their investment for maximum profit.

2. Engineering and Science

- **Electrical Circuits (Kirchhoff's Laws):** In circuit analysis, multiple loops with resistors and voltage sources create simultaneous equations that engineers solve to find current values.
- **Mixture Problems in Chemistry:** Determining the correct proportions of two solutions with different concentrations to get a desired mixture. Example:

  $x + y = 100$ (Total volume of solution)
  $0.2x + 0.5y = 0.35(100)$ (Total concentration equation

3. Agriculture and Farming

- **Crop Planning:** Farmers may use equations to decide how much land to allocate for different crops while considering constraints like water supply and cost.
- **Animal Feed Optimization:** Determining the right mix of grains and proteins for livestock based on nutritional requirements.

Exercise - 4

1. Solve each of the following systems by graphing:

   (a) $x - 2y = 2$, $x + y = 5$

   (b) $x + 2y = -4$, $2x + 4y = 8$

   (c) $2x + 4y = 8$, $x + 2y = 4$

2. Solve by substitution:

$$3x + 2y = -2$$
$$2x - y = -6$$

3. Jasmine wants to use milk and orange juice to increase the amount of calcium and vitamin A in her daily diet. An ounce of milk contains 37 milligrams of calcium and 57 micrograms of vitamin A. An ounce of orange juice contains 5 milligrams of calcium and 65 micrograms of vitamin A. How many ounces of milk and orange juice should Jasmine drink each day to provide exactly 500 milligrams of calcium and 1,200 micrograms of vitamin A?

4. Solve, by determinants, the following set of simultaneous equations

$$5x - 6y + 4z = 15$$
$$7x + 4y - 3z = 19$$
$$2x + y + 6z = 46$$

5. Solve the following system of equations using Cramer's Rule:

$$2x - 3y + 4z = -9$$
$$-3x + 4y + 2 = -12$$
$$4x - 2y - 3z = -3$$

6. Solve the following equations by matrix inversion method:

$$x + y + z = 4$$
$$2x - y + 3z = 1$$
$$3x + 2y - z = 1$$

7. Find all solutions for the linear system

$$2x + y - z = 1$$
$$x + 3y + 4z = 2$$
$$7x + 3y + 4z = 1$$

Chapter 5

Eigen Values and Eigen Vectors

5.1 Introduction

From an applications viewpoint, eigenvalue problems are probably the most important problems that arise in connection with matrix analysis. In this Chapter, we discuss the basic concepts of eigen values and eigen vectors. We shall see that eigenvalues and eigenvectors are associated with square matrices of order $n \times n$. If n is small (2 or 3), determining eigenvalues is a fairly straightforward process (requiring the solution of a low order polynomial equation). In linear algebra, an eigenvector or characteristic vector is a vector that has its direction unchanged (or reversed) by a given linear transformation. More precisely, an eigenvector, v of a linear transformation, T is scaled by a constant factor, λ, when the linear transformation is applied to it: $Tv = \lambda v$ The corresponding eigenvalue, characteristic value, or characteristic root is the multiplying factor λ (possibly negative).

Geometrically, vectors are multi-dimensional quantities with magnitude and direction, often pictured as arrows. A linear transformation rotates, stretches, or shears the vectors upon which it acts. Its eigenvectors are those vectors that are only stretched, with neither rotation nor shear. The corresponding eigenvalue is the factor by which an eigenvector is stretched or squished. If the eigenvalue is negative, the eigenvector's direction is reversed.

The eigenvectors and eigenvalues of a linear transformation serve to characterize it, and so they play important roles in all the areas where linear algebra is applied, from geology to quantum mechanics. In particular, it is often the case that a system is represented by a linear transformation whose outputs are fed as inputs to the same transformation (feedback). In such an application, the largest eigenvalue is of particular importance, because it governs the long-term behaviors of the system after many applications of the linear transformation, and the associated eigenvector is the steady state of the system.

5.2 Definition

Consider the linear transformation of n-dimensional vectors defined by an $n \times n$ matrix A as

$$Av = w,$$

where

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \dots \\ v_n \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \dots \\ w_n \end{bmatrix}.$$

For each row,

$$w_i = A_{i1}v_1 + A_{i2}v_2 + A_{i3}v_3 + \cdots + A_{in}v_n.$$

If it occurs that *v* and *w* are scalar multiples, that is if

$$Av = w = \lambda v \qquad (1)$$

then v is an **eigenvector** of the linear transformation A and the scale factor λ is the **eigenvalue** corresponding to that eigenvector. Equation (1) is the **eigenvalue equation** for the matrix A.

Equation (1) can be stated equivalently as

$$(A - \lambda I)v = 0, \qquad (2)$$

where I is the $n \times n$ identity matrix and 0 is the zero vector.

Equation (2) has a nonzero solution v if and only if the determinant of the matrix $(A - \lambda I)$ is zero. Therefore, the eigenvalues of A are values of λ that satisfy the equation

$$|A - \lambda I| = 0. \qquad (3)$$

Using the Leibniz formula for determinants, the left-hand side of equation (3) is a polynomial function of the variable λ and the degree of this polynomial is n, the order of the matrix A. Its coefficients depend on the entries of A, except that its term of degree n is always $(-1)^n\lambda^n$. This polynomial is called the **characteristic polynomial** of A. Equation (3) is called the **characteristic equation** or the secular equation of A.

The fundamental theorem of algebra implies that the characteristic polynomial of an $n \times n$ matrix A, being a polynomial of degree n, can be factored into the product of n linear terms,

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \ldots (\lambda_n - \lambda). \qquad (4)$$

where each λ_i may be real but in general is a complex number. The numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, which may not all have distinct values, are roots of the polynomial and are the eigenvalues of A.

If A is any square matrix of order n, we can form the matrix A - λI, where I is the nth order unit matrix. The determinant of this matrix equated to zero,

that is, $|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} - \lambda & \ldots & a_{1n} - \lambda \\ a_{21} - \lambda & a_{22} - \lambda & \ldots & a_{2n} - \lambda \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} - \lambda & a_{n2} - \lambda & \ldots & a_n - \lambda \end{vmatrix} = 0,$

is called the ***characteristic equation* of A**.

The roots of this equation are called the ***eigenvalues or characteristic roots*** of the matrix A.

If X = $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \dots \\ x_n \end{pmatrix}$ and A = $\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_n \end{pmatrix}$, then corresponding to the eigen value λ we may write

$$\lambda X = AX$$

and X is called the ***eigen vector*** corresponding to the eigen value λ.

As a brief example, which is described in more detail in the examples section later, consider the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Taking the determinant of $(A - \lambda I)$, the characteristic polynomial of A is

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 3 - 4\lambda + \lambda^2.$$

Setting the characteristic polynomial equal to zero, we obtain the characteristic equation. It has roots at $\lambda=1$ and $\lambda=3$, which are the two eigenvalues of A. The eigenvectors corresponding to each eigenvalue can be found by solving for the components of v in the equation

$$(A - \lambda I)v = 0.$$

From the above equation it is seen that the eigenvectors are any nonzero scalar multiples of

$$v_{\lambda=1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ and } v_{\lambda=3} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

If the entries of the matrix A are all real numbers, then the coefficients of the characteristic polynomial will also be real numbers, but the eigenvalues may still have nonzero imaginary parts. The entries of the corresponding eigenvectors therefore may also have nonzero imaginary parts. Similarly, the eigenvalues may be irrational numbers even if all the entries of A are rational numbers or even if they are all integers. However, if the entries of A are all algebraic numbers, which include the rational, the eigenvalues must also be algebraic numbers.

The non-real roots of a real polynomial with real coefficients can be grouped into pairs of complex conjugates, namely with the two members of each pair having imaginary parts that differ only in sign and the same real part. If the degree is odd, then by the intermediate value theorem at least one of the roots is real. Therefore, any real matrix with odd order has at least one real eigenvalue, whereas a real matrix with even order may not have any real eigenvalues. The eigenvectors associated with these complex eigenvalues are also complex and also appear in complex conjugate pairs.

5.3 Spectrum of a matrix

The **spectrum** of a matrix is the list of eigenvalues, repeated according to multiplicity; in an alternative notation the set of eigenvalues with their multiplicities. An important quantity associated with the spectrum is the maximum absolute value of any eigenvalue. This is known as the **spectral radius** of the matrix.

5.4 Algebraic multiplicity

Let λ_i be an eigenvalue of an $n \times n$ matrix A. The algebraic multiplicity $\mu_A(\lambda_i)$ of the eigenvalue is its multiplicity as a root of the characteristic polynomial, that is, the largest integer k such that $(\lambda - \lambda_i)^k$ divides evenly that polynomial.

Suppose a matrix A has dimension n and $d \leq n$ distinct eigenvalues. Whereas equation (4) factors the characteristic polynomial of A into the product of n linear terms with some terms potentially repeating, the characteristic polynomial can also be written as the product of d terms each corresponding to a distinct eigenvalue and raised to the power of the algebraic multiplicity,

$$\det(A - \lambda I) = (\lambda_1 - \lambda)^{\mu_A(\lambda_1)}(\lambda_2 - \lambda)^{\mu_A(\lambda_2)} \dots (\lambda_n - \lambda)^{\mu_A(\lambda_n)}.$$

If $d = n$ then the right-hand side is the product of n linear terms and this is the same as equation(4). The size of each eigenvalue's algebraic multiplicity is related to the dimension n as

$$1 \leq \mu_A(\lambda_i) \leq n$$

and $\mu_A = \sum_{i=1}^{d} \mu_A(\lambda_i) = n.$

If $\mu_A(\lambda_i) = 1$, then λ_i is said to be a simple eigenvalue. If $\mu_A(\lambda_i)$ equals the geometric multiplicity of λ_i , $\gamma_A(\lambda_i)$, defined in the next section, then λ_i is said to be a semisimple eigenvalue.

5.5 Eigenspaces and geometric multiplicity

Given a particular eigenvalue λ of the $n \times n$ matrix A, define the set E to be all vectors v that satisfy equation (2)

$$E = \{v: (A - \lambda I)v = 0\}.$$

On one hand, this set is precisely the kernel or null space of the matrix $(A - \lambda I)$.

On the other hand, by definition, any nonzero vector that satisfies this condition is an eigenvector of A associated with λ. So, the set E is the union of the zero vector with the set of all eigenvectors of A associated with λ, and E equals the null space of $(A - \lambda I)$. E is called the **eigenspace** or **characteristic space** of A associated with λ. In general, λ is a complex number and the eigenvectors are complex n by 1 matrices. A property of the null space is that it is a linear subspace, so E is a linear subspace of $\mathbb{C}^n$.

Because the eigenspace E is a linear subspace, it is closed under addition. That is, if two vectors **u** and **v** belong to the set E, written **u**, **v** ∈ E, then

$$(\mathbf{u} + \mathbf{v}) \in E \text{ or equivalently } A(\mathbf{u} + \mathbf{v}) = \lambda(\mathbf{u} + \mathbf{v}).$$

This can be checked using the distributive property of matrix multiplication. Similarly, because E is a linear subspace, it is closed under scalar multiplication. That is, if **v** ∈ E and α is a complex number,

$$(\alpha\mathbf{v}) \in E \text{ or equivalently } A(\alpha\mathbf{v}) = \lambda(\alpha\mathbf{v}).$$

This can be checked by noting that multiplication of complex matrices by complex numbers is commutative. As long as **u** + **v** and α**v** are not zero, they are also eigenvectors of A associated with λ.

The dimension of the eigenspace E associated with λ, or equivalently the maximum number of linearly independent eigenvectors associated with λ, is referred to as the eigenvalue's **geometric multiplicity** $\gamma_A(\lambda)$. Because E is also the nullspace of $(A - \lambda I)$, the geometric multiplicity of λ is the dimension of the nullspace of $(A - \lambda I)$, also called the *nullity* of $(A - \lambda I)$, which relates to the dimension and rank of $(A - \lambda I)$ as

$$\gamma_A(\lambda) = n - rank(A - \lambda I).$$

Because of the definition of eigenvalues and eigenvectors, an eigenvalue's geometric multiplicity must be at least one, that is, each eigenvalue has at least one associated eigenvector. Furthermore, an eigenvalue's geometric multiplicity cannot exceed its algebraic multiplicity.

5.6 Properties of Eigenvalues

Let A be an arbitrary $n \times n$ matrix of complex numbers with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Each eigenvalue appears $\mu_A(\lambda_i)$ times in this list, where $\mu_A(\lambda_i)$ is the eigenvalue's algebraic multiplicity. The following are properties of this matrix and its eigenvalues:

1. The trace of A, defined as the sum of its diagonal elements, is also the sum of all eigenvalues.
2. The determinant of A is the product of all its eigenvalues.
3. The eigenvalues of the k^{th} power of A; i.e., the eigenvalues of A^k , for any positive integer k, are $\lambda_1{}^k, \lambda_2{}^k, \ldots, \lambda_n{}^k$.

The matrix A is invertible if and only if every eigenvalue is nonzero.

1. If A is invertible, then the eigenvalues A^{-1} of are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$ and each eigenvalue's geometric multiplicity coincides. Moreover, since the characteristic polynomial of the inverse is the reciprocal polynomial of the original, the eigenvalues share the same algebraic multiplicity.
2. If A is equal to its conjugate transpose A^* , or equivalently if is Hermitian, then every eigenvalue is real. The same is true of any symmetric real matrix.
3. If A is not only Hermitian but also positive-definite, positive-semidefinite, negative-definite, or negative-semidefinite, then every eigenvalue is positive, non-negative, negative, or non-positive, respectively.
4. If A is unitary, every eigenvalue has absolute value ⍰$|\lambda_i| = 1$.
5. If A is a $n \times n$ matrix and are its eigenvalues, then the eigenvalues of matrix I+A (where I is the identity matrix) are $\{\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_k + 1\}$. Moreover, if , $\alpha \in \mathbb{C}$ the eigenvalues of $\alpha I + A$ are $\{\lambda_1 + \alpha, \lambda_2 + \alpha, \ldots, \lambda_k + \alpha\}$. . More generally, for a polynomial P the eigenvalues of matrix P(A) are $\{\mathrm{P}(\lambda_1),\ \mathrm{P}(\lambda_2), \ldots, \mathrm{P}(\lambda_k)\}$.
6. The Eigen vectors correspond to distinct Eigen values of a matrix are linearly independent.
7. The Eigen values of a symmetric matrix the Eigen values are either zero (or) purely imaginary.
8. The Eigen values of an orthogonal matrix are of unit modulus i. e. $|\lambda| = 1$.

5.7 Left and right eigenvectors

Many disciplines traditionally represent vectors as matrices with a single column rather than as matrices with a single row. For that reason, the word "eigenvector" in the context of matrices almost always refers to a **right eigenvector**, namely a *column* vector that *right* multiplies the $n \times n$ matrix A in the defining equation, equation (1),

$$Av = \lambda v.$$

The eigenvalue and eigenvector problem can also be defined for *row* vectors that *left* multiply matrix A. In this formulation, the defining equation is

$$uA = \kappa u,$$

where k is a scalar and u is a $1 \times n$ matrix. Any row vector u satisfying this equation is called a **left eigenvector** of A and k is its associated eigenvalue. Taking the transpose of this equation,

$$A^T u^T = k u^T.$$

Comparing this equation to equation (1), it follows immediately that a left eigenvector of A is the same as the transpose of a right eigenvector of A^T, with the same eigenvalue. Furthermore, since the characteristic polynomial of A^T is the same as the characteristic polynomial of A, the left and right eigenvectors of A are associated with the same eigenvalues.

5.8 Cayley-Hamilton Theorem

In Linear Algebra, the Cayley–Hamilton theorem (named after the mathematicians Arthur Cayley and William Rowan Hamilton) states that every square matrix over a commutative ring (such as the real or complex numbers or the integers) satisfies its own characteristic equation.

The characteristic polynomial of an $n \times n$ matrix A is defined as

$$p_A(\lambda) = \det(\lambda I_n - A),$$

where det is the determinant operation, λ is a variable scalar element of the base ring and I_n is the $n \times n$ identity matrix. Since each entry of the matrix $(\lambda I_n - A)$ is either constant or linear in λ. So, it can be written as

$$p_A(\lambda) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0.$$

By replacing the scalar variable λ with the matrix A, one can define an analogous matrix polynomial expression,

$$p_A(A) = A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0.$$

Here, A is the given matrix—not a variable, unlike λ—so $p_A(A)$ is a constant rather than a function.) The Cayley–Hamilton theorem states that this polynomial expression is equal to the zero matrix, which is to say that

$$p_A(A) = 0,$$

that is, the characteristic polynomial p_A is an annihilating polynomial for A. One use for the Cayley–Hamilton theorem is that it allows A^n to be expressed as a linear combination of the lower matrix powers of A:

$$A^n = -c_{n-1}A^{n-1} - \cdots - c_1 A - c_0.$$

When the ring is a field, the Cayley–Hamilton theorem is equivalent to the statement that the minimal polynomial of a square matrix divides its characteristic polynomial.

5.9 Illustrative examples:

Example 1: Find the eigen values and eigen vectors of the matrix $\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$.

Solution: Given A = $\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$ (say)

The characteristic equation corresponding to the eigen value λ is given by $|A - \lambda I\,| = 0$

that is, $\begin{vmatrix} 5 & 4 \\ 1 & 2 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 5 & 4 \\ 1 & 2 \end{vmatrix} - \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\text{or, } (5-\lambda)(2-\lambda)-4=0$$
$$\text{or, } \lambda^2-7\lambda+6=0$$
$$\text{or, } \lambda^2-6\lambda-\lambda+6=0$$
$$\text{or, } (\lambda-6)(\lambda-2)=0,$$
$$\text{or, } \lambda=6 \text{ and } 1.$$

Thus, the eigen values are 6 and 1.

Let X = $\begin{pmatrix} x \\ y \end{pmatrix}$ be an eigen vector corresponding to the eigenvalue λ, such that

$$AX = \lambda X$$

Corresponding to $\lambda = 6$,

$$\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 6\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 5x+4y \\ x+2y \end{pmatrix} = \begin{pmatrix} 6x \\ 6y \end{pmatrix}$$

⇒ 5x + 4y = 6x and x + 2y = 6y

⇒ -x + 4y = 0 and x – 4y = 0

Let y = k, k is any real number.
Then x = 4k.
Thus, X = $\begin{pmatrix} 4k \\ k \end{pmatrix} = k\begin{pmatrix} 4 \\ 1 \end{pmatrix}$.
Thus, $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is the eigen vector corresponding to the eigen value 6.
Corresponding to $\lambda = 1$,

$$\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 5x+4y \\ x+2y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

⇒ 5x + 4y = x and x + 2y = y

⇒ 4x + 4y = 0 and x + y = 0

Let y = k, k is any real number.
Then x = -k.
Thus, X = $\begin{pmatrix} -k \\ k \end{pmatrix} = k\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.
Thus, $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is the eigen vector corresponding to the eigen value 1.

Example 2: Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 1 & -2 \\ -6 & 0 \end{pmatrix}$.

Solution: Given A = $\begin{pmatrix} 1 & -2 \\ -6 & 0 \end{pmatrix}$ (say)
The characteristic equation corresponding to the eigen value λ is given by $|A-\lambda I\ |=0$

That is, $\begin{vmatrix} 1 & -2 \\ -6 & 0 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1 & -2 \\ -6 & 0 \end{vmatrix} - \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -2 \\ -6 & 0-\lambda \end{vmatrix} = 0$$

or, $(1-\lambda)(-\lambda) - 12 = 0$

or, $\lambda^2 - \lambda - 12 = 0$

or, $(\lambda - 4)(\lambda + 3) = 0,$

or, $\lambda = 4 \text{ and} - 3.$

Thus, the eigen values are 4 and -3.

Let X = $\begin{pmatrix} x \\ y \end{pmatrix}$ be an eigen vector corresponding to the eigenvalue λ, such that

$$AX = \lambda X$$

Corresponding to $\lambda = 4,$

$$\begin{pmatrix} 1 & -2 \\ -6 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 4 \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x - 2y \\ -6x + 0 \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \end{pmatrix}$$

⇒ x - 2y = 4x and -6x = 4y

⇒ 3x +2y = 0 and 3x + 2y = 0

Let x = k, k is any real number.

Then y = $-\frac{3k}{2}$.

Thus, X = $\begin{pmatrix} k \\ -\frac{3k}{2} \end{pmatrix} = k \begin{pmatrix} 1 \\ -\frac{3}{2} \end{pmatrix}$.

Thus, $\begin{pmatrix} 1 \\ -\frac{3}{2} \end{pmatrix}$ is the eigen vector corresponding to the eigen value 4.

Corresponding to $\lambda = -3,$

$$\begin{pmatrix} 1 & -2 \\ -6 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -3 \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x - 2y \\ -6x + 0 \end{pmatrix} = \begin{pmatrix} -3x \\ -3y \end{pmatrix}$$

⇒ x - 2y = -3x and -6x = -3y

⇒ 2x = y and 2x = y

Let x = k, k is any real number.

Then y =2k .

Thus, X = $\begin{pmatrix} k \\ 2k \end{pmatrix} = k\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Thus, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is the eigen vector corresponding to the eigen value -3.

Example 3: Find the eigen values and eigen vectors of the matrix

$$\begin{pmatrix} \mathbf{3} & \mathbf{1} & \mathbf{4} \\ \mathbf{0} & \mathbf{2} & \mathbf{6} \\ \mathbf{0} & \mathbf{0} & \mathbf{5} \end{pmatrix}.$$

Solution: The characteristic equation is corresponding to the eigen value λ is given by

$$|A - \lambda I\ | = 0$$

$$\Rightarrow \begin{vmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)(2-\lambda)(\ 5-\lambda) = 0$$

Thus, the eigen values are 2, 3 and 5.

Let X = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be an eigen vector corresponding to the eigenvalue λ, such that

$$AX = \lambda X.$$

Now, corresponding to the eigen value 2 we have,

$$\begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 3x + y + 4z \\ 2y + 6z \\ 5z \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

$$\Rightarrow 3x + y + 4z = 2x$$

$$2y + 6z = 2y \text{ and } 5z = 2z$$

$$\Rightarrow x + y + 4z = 0$$
$$6z = 0$$
$$3z = 0$$

$$\Rightarrow z\ = 0.$$

Let x = k, k is any real number. Then y = -1

Thus, X = $\begin{pmatrix} k \\ -k \\ 0 \end{pmatrix} = k\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

Thus the eigen vector corresponding to the eigen value 2 is $\begin{pmatrix}1\\-1\\0\end{pmatrix}$. And $\begin{pmatrix}1\\-1\\0\end{pmatrix}$

In a similar way, the eigen vectors corresponding to the eigen values 3 and 5 are $\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}3\\2\\1\end{pmatrix}$ respectively.

Example 4: Find the eigen values and eigen vectors of the matrix $\begin{pmatrix}\mathbf{1} & \mathbf{1} & \mathbf{3}\\ \mathbf{1} & \mathbf{5} & \mathbf{1}\\ \mathbf{3} & \mathbf{1} & \mathbf{1}\end{pmatrix}$.

Solution: The characteristic equation is corresponding to the eigen value λ is given by

$$|A - \lambda I\ | = 0$$

$$\Rightarrow \begin{vmatrix}1 & 1 & 3\\ 1 & 5 & 1\\ 3 & 1 & 1\end{vmatrix} - \lambda \begin{vmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix}1-\lambda & 1 & 3\\ 1 & 5-\lambda & 1\\ 3 & 1 & 1-\lambda\end{vmatrix} = 0$$

$$\Rightarrow (\lambda - 3)(\lambda - 6)(\ \lambda + 2) = 0$$

Thus, the eigen values are 3, 6 and -2.

Let X = $\begin{pmatrix}x\\y\\z\end{pmatrix}$ be an eigen vector corresponding to the eigenvalue λ, such that

AX = λX.

Now, corresponding to the eigen value -2 we have,

$$\begin{pmatrix}1 & 1 & 3\\ 1 & 5 & 1\\ 3 & 1 & 1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = -2\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}x + y + 3z\\ x + 5y + z\\ 3x + y + z\end{pmatrix} = \begin{pmatrix}-2x\\ -\ 2y\\ -2z\end{pmatrix}$$

$$\Rightarrow x + y + 3z = -2x$$
$$x + 5y + z = -2y$$
$$\text{and } 3x + y + z = -2z$$

Solving, $y = 0$.

Let x = k, k is any real number. Then z = -k.

Thus, X = $\begin{pmatrix} k \\ 0 \\ -k \end{pmatrix} = k\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

Let X = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be an eigen vector corresponding to the eigenvalue λ, such that

$$AX = \lambda X.$$

Now, corresponding to the eigen value 3 we have,

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x+y+3z \\ x+5y+z \\ 3x+y+z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix}$$

$$\Rightarrow x+y+3z = 3x$$
$$x+5y+z = 3y$$
$$\text{and } 3x+y+z = 3z$$

Solving, $x = -y = z$.

Let x = k, k is any real number. Then y = -k and z = k

Thus, X = $\begin{pmatrix} k \\ -k \\ k \end{pmatrix} = k\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

Let X = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be an eigen vector corresponding to the eigenvalue λ, such that

$$AX = \lambda X.$$

Now, corresponding to the eigen value 6 we have,

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 6\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x+y+3z \\ x+5y+z \\ 3x+y+z \end{pmatrix} = \begin{pmatrix} 6x \\ 6y \\ 6z \end{pmatrix}$$

$$\Rightarrow x+y+3z = 6x$$
$$x+5y+z = 6y$$
$$\text{and } 3x+y+z = 6z$$

Solving, $x = z = \frac{y}{2}$.

Let y = k, k is any real number. Then x = z = 2k.

Thus, X = $\begin{pmatrix} 2k \\ k \\ 2k \end{pmatrix} = k\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$.

Example 5: Find the eigen values and eigen vectors of the matrix $\begin{pmatrix} \mathbf{5} & \mathbf{-10} & \mathbf{-5} \\ \mathbf{2} & \mathbf{14} & \mathbf{2} \\ \mathbf{-4} & \mathbf{-8} & \mathbf{6} \end{pmatrix}$.

Solution: The characteristic equation is corresponding to the eigen value λ is given by

$$|A - \lambda I\ | = 0$$

$$\Rightarrow \begin{vmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 5-\lambda & -10 & -5 \\ 2 & 14-\lambda & 2 \\ -4 & -8 & 6-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (\lambda - 5)(\lambda - 10)^2 = 0$$

Thus, the eigen values are 5, 10 and 10.

Let X = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be an eigen vector corresponding to the eigenvalue λ, such that

$$\text{AX} = \lambda X.$$

Now, corresponding to the eigen value 5 we have,

$$\begin{pmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 5\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

$$\Rightarrow \begin{pmatrix} 5x - 10y - 5z \\ 2x + 14y + 2z \\ -4x - 8y + 6z \end{pmatrix} = \begin{pmatrix} 5x \\ 5y \\ 5z \end{pmatrix}$$

$$\Rightarrow -10y - 5z = 0$$
$$2x + 9y + 2z = 0$$
$$\text{and } -4x - 8y + z = 0$$

Solving, we get the eigenvector corresponding to the eigenvalue 5 as $\begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix}$.

Let X = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be an eigen vector corresponding to the eigenvalue λ, such that

$$\text{AX} = \lambda X.$$

Now, corresponding to the eigen value 10 we have,

$$\begin{pmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 10\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 5x - 10y - 5z \\ 2x + 14y + 2z \\ -4x - 8y + 6z \end{pmatrix} = \begin{pmatrix} 10x \\ 10y \\ 10z \end{pmatrix}$$

$$\Rightarrow 5x - 10y - 5z = 10x$$
$$2x + 14y + 2z = 10y$$
$$\text{and } -4x - 8y + 6z = 10z$$
$$\Rightarrow x + 2y + z = 0.$$

So, the eigenvectors are of the form

$$\begin{pmatrix} 2s - t \\ s \\ t \end{pmatrix} = s\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ are two eigenvectors corresponding to the eigenvalue 10.

Example 6: Find the eigen values and eigen vectors of the matrix
$$\begin{pmatrix} \mathbf{2} & \mathbf{2} & \mathbf{2} \\ \mathbf{2} & \mathbf{2} & \mathbf{2} \\ \mathbf{2} & \mathbf{2} & \mathbf{2} \end{pmatrix}.$$

Solution: The characteristic equation is corresponding to the eigen value λ is given by

$$|A - \lambda I\ | = 0$$

$$\Rightarrow \begin{vmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2 - \lambda & 2 & 2 \\ 2 & 2 - \lambda & 2 \\ 2 & 2 & 2 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (\lambda - 5)(\lambda - 10)^2 = 0$$

Thus, the eigen values are 0, 0 and 6.

Let X = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be an eigen vector corresponding to the eigenvalue λ, such that

$$\text{AX} = \lambda X.$$

Now, corresponding to the eigen value 0 we have,

$$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow x + y + z = 0.$$

Let $y = k_1$ and $z = k_2$, then $x = -(k_1 + k_2)$.

Then X = $\begin{pmatrix} -(k_1 + k_2) \\ k_1 \\ k_2 \end{pmatrix} = k_1\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + k_2\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

Thus, two eigenvectors corresponding to the eigenvalue 0 are $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

Now, corresponding to the eigen value 6 we have,

$$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 6\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow -2x + y + z = 0$$
$$x - 2y + z = 0$$
$$x + y - 2z = 0.$$

Let $y = z = k$. Then $x = k$, that is, $x = y = z = k$.

Then X = $\begin{pmatrix} k \\ k \\ k \end{pmatrix} = k\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Thus, two eigenvectors corresponding to the eigenvalue 6 is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Example 7: Find the eigen values and eigen vectors of the matrix $\begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{3} \end{pmatrix}$.

Solution: The characteristic equation is corresponding to the eigen value λ is given by

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} - \lambda\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda)(3-\lambda) = 0$$

Thus, the eigen values are 1, 2 and 3.

Let X = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be an eigen vector corresponding to the eigenvalue λ, such that

AX = λX.

Now, corresponding to the eigen value 1 we have,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow x = x, 2y = y \text{ and } 3z = z.$$

Let $x = k$, where k is any non-zero real number.

We get, $y = 0$ and $z = 0$.

Then X = $\begin{pmatrix} k \\ 0 \\ 0 \end{pmatrix} = k\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Therefore, the eigenvector corresponding to the eigenvalue 1 is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Now, corresponding to the eigen value 2 we have,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow x = 2x, 2y = 2y \text{ and } 3z = 2z.$$

Let $y = k$, where k is any non-zero real number.

We get, $x = 0$ and $z = 0$.

Then X = $\begin{pmatrix} 0 \\ k \\ 0 \end{pmatrix} = k\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Therefore, the eigenvector corresponding to the eigenvalue 2 is $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Now, corresponding to the eigen value 3 we have,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow x = 3x, 2y = 3y \text{ and } 3z = 3z.$$

Let $z = k$, where k is any non-zero real number.

We get, $x = 0$ and $y = 0$.

Then X = $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = k\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Therefore, the eigenvector corresponding to the eigenvalue 3 is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Example 8: Find the eigen values and eigen vectors of the matrix

$$\begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{4} \\ \mathbf{0} & \mathbf{4} & \mathbf{7} \\ \mathbf{0} & \mathbf{0} & \mathbf{6} \end{pmatrix}.$$

Solution: The characteristic equation is corresponding to the eigen value λ is given by

$$|A - \lambda I\,| = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 2 & 4 \\ 0 & 4 & 7 \\ 0 & 0 & 6 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 4 \\ 0 & 4-\lambda & 7 \\ 0 & 0 & 6-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(4-\lambda)(6-\lambda) = 0$$

Thus, the eigen values are 1, 4 and 6.

Now, corresponding to the eigen value 1 we have,

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 4 & 7 \\ 0 & 0 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow y + 2z = 0$$
$$3y + 7z = 0$$
$$5z = 0$$

Thus, we get $z = 0$.

Let, $x = k$ and $y = -\frac{5k}{2}$.

Then X = $\begin{pmatrix} k \\ -\frac{5k}{2} \\ 0 \end{pmatrix} = k\begin{pmatrix} 1 \\ -\frac{5}{2} \\ 0 \end{pmatrix} = k\begin{pmatrix} 2 \\ -5 \\ 0 \end{pmatrix}$.

Therefore, the eigenvector corresponding to the eigenvalue 1 is $\begin{pmatrix} 2 \\ -5 \\ 0 \end{pmatrix}$.

Now, corresponding to the eigen value 4 we have,

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 4 & 7 \\ 0 & 0 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow -3x + 2y + 4z = 0$$
$$7z = 0$$
$$2z = 0$$

Thus, $z = 0$.
Let, $x = k$ and thus, $y = \frac{3}{2}k$.

Then X = $\begin{pmatrix} k \\ \frac{3k}{2} \\ 0 \end{pmatrix} = k\begin{pmatrix} 1 \\ \frac{3}{2} \\ 0 \end{pmatrix} = k\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$.

Therefore, the eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$.

Now, corresponding to the eigen value 6 we have,

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 4 & 7 \\ 0 & 0 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 6\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow -5x + 2y + 4z = 0$$
$$-2y + 7z = 0$$
$$z = 0$$

Thus, $z = k$.
Let, $x = \frac{11k}{5}$ and thus, $y = \frac{7k}{2}$.

Then X = $\begin{pmatrix} \frac{11k}{5} \\ \frac{7k}{2} \\ 0 \end{pmatrix} = k\begin{pmatrix} \frac{11}{5} \\ \frac{7}{2} \\ 0 \end{pmatrix} = k\begin{pmatrix} 22 \\ 35 \\ 0 \end{pmatrix}$.

Therefore, the eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix} 22 \\ 35 \\ 0 \end{pmatrix}$.

Example 9: Verify Cayley-Hamilton theorem for the matrix A = $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ and find its inverse. Also express $A^5 - 4A^4 - 7A^3 + 11A^2$-A-10 I as a linear polynomial in A.

Solution: The characteristic equation corresponding to the eigen value λ is given by $|A - \lambda I\,| = 0$

That is, $\begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} - \lambda\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} - \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(\,3-\lambda) - 8 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda - 5 = 0$$

By Cayley-Hamilton theorem, A must satisfy its characteristic equation, so that we have to verify that if

$$A^2 - 4A - 5I = O. \qquad \text{………………..(1)}$$

Now, $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - 4\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 9 & 16 \\ 8 & 17 \end{pmatrix} - \begin{pmatrix} 4 & 16 \\ 8 & 12 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ = O.

Thus, the theorem is verified.

Now, multiplying equation (1) by A^{-1}, we get

$$A^{-1}(A^2 - 4A - 5I) = O$$

$$\Rightarrow A - 4I - 5A^{-1} = 0$$

$$\Rightarrow \; A^{-1} = \frac{1}{5}(A - 4I)$$

$$= \frac{1}{5}\left\{\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}$$

$$= \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 2 & -1 \end{pmatrix}$$

Now dividing the polynomial $\lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2$-A-10 I by the polynomial $\lambda^2 - 4\lambda - 5$, we obtain

$\lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2$-A-10 I

$= (\lambda^2 - 4\lambda - 5)(\lambda^3 - 2\lambda + 3) + \lambda + 5$

$= \lambda + 5$

Hence, $A^5 - 4A^4 - 7A^3 + 11A^2$-A-10 I= A + 5I, is a linear polynomial in A.

Example 10: Find the characteristic equation of the matrix A = $\begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{3} \\ \mathbf{1} & \mathbf{3} & \mathbf{-3} \\ \mathbf{-2} & \mathbf{-4} & \mathbf{-4} \end{pmatrix}$ **and hence find its inverse.**

Solution: The characteristic equation is $\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 3-\lambda & -3 \\ -2 & -4 & -4-\lambda \end{vmatrix} = 0$

$$\text{or}, (1-\lambda)\{(3-\lambda)(-4-\lambda) - 12\} - 1\{(-4-\lambda)\} + 3\{-4 + 2(3-\lambda)\} = 0$$

$$\text{or}, \lambda^3 - 20\lambda + 8 = 0$$

Thus, by Cayley-Hamilton theorem, $A^3 - 20A + 8I = O$.

Multiplying both sides by A^{-1}, $\quad A^{-1}A^3 - 20A^{-1}A + 8A^{-1}I = O$

$$\Rightarrow A^2 - 20I + 8A^{-1} = O$$

$$\Rightarrow A^{-1} = \frac{5}{2}I - \frac{1}{8}A^2$$

$$= \frac{5}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{8}\begin{pmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{pmatrix}.$$

5.10 Matrix Diagonalization:

Matrix diagonalization is the process of reducing a square matrix into its diagonal form using a similarity transformation. This process is useful because diagonal matrices are easier to work with, especially when raising them into integer powers.

Not all matrices are diagonalizable. A matrix is diagonalizable if it has no defective eigenvalues, meaning each eigenvalue's geometric multiplicity is equal to its algebraic multiplicity.

Matrix similarity transformation:

Let A and B be two matrices of order n. Matrix B is considered similar to A if there exists an invertible matrix P such that:

$$\mathbf{B} = \mathbf{P^{-1} A\, P}$$

This transformation is known as Matrix similarity transformation. Similar matrices have the same rank, trace, determinant, and eigenvalues.

Diagonalization of a matrix:

Diagonalization of a matrix refers to the process of transforming any matrix A into its diagonal form D. According to the similarity transformation, if A is diagonalizable, then

$$\mathbf{D} = \mathbf{P^{-1} A\, P}$$

Where D is a **diagonal matrix** and P is **modal matrix**.

A modal matrix is an $n \times n$ matrix consisting of the eigenvectors of A. It is essential in the process of diagonalization and similarity transformation.

Conditions for diagonalization:

A matrix is diagonalizable if it has n linearly independent eigenvectors, or if the sum of the geometric multiplicities of its eigenvalues is n.

Example 1: Diagonalize the matrix $\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$.

Solution: The characteristic equation is corresponding to the eigen value λ is given by

$$|A - \lambda I\,| = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (\lambda-1)(\lambda-2)(\lambda-3) = 0$$

Thus, the eigen values are 1, 2 and 3.

Now, corresponding to the eigen value 1 we have,

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Then X = $\begin{pmatrix} k \\ -k \\ 0 \end{pmatrix} = k\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

Therefore, the eigenvector corresponding to the eigenvalue 1 is $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

Similarly, for the eigenvalues 2 and 3, we get the eigenvectors as $\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$.

Thus, we may write the modal matrix as $P = \begin{pmatrix} 1 & -2 & 1 \\ -1 & 1 & -1 \\ 0 & 2 & -2 \end{pmatrix}$.

We get $|P| = \begin{vmatrix} 1 & -2 & 1 \\ -1 & 1 & -1 \\ 0 & 2 & -2 \end{vmatrix} = 2 \neq 0$

Therefore, $P^{-1} = \frac{1}{2}\begin{pmatrix} 0 & -2 & 1 \\ -2 & -2 & 0 \\ -2 & -2 & -1 \end{pmatrix}$.

Thus, we get the Diagonal matrix

$$D = P^{-1}AP = \frac{1}{2}\begin{pmatrix} 0 & -2 & 1 \\ -2 & -2 & 0 \\ -2 & -2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & -2 & 1 \\ -1 & 1 & -1 \\ 0 & 2 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

5.11 Applications of Eigenvalues and Eigenvectors in different fields:

Originally eigenvalues and eigenvectors were used to study principal axes of the rotational motion of rigid bodies, but now widely used in stability analysis, atomic orbitals, facial recognition, and matrix diagonalization.

Also, it is used in geometric transformations, principal component analysis, in graph theory, Markov chain analysis, vibration analysis, stress and strain analysis, wave transport, molecular orbitals, geology and glaciology, basic reproduction number, eigenfaces etc.

Exercises:

1. Find the eigenvectors of the matrix A = $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.
2. Find the eigenvalues and eigenvectors of A, A^2, $A + 4I$ and A^{-1} of $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Also, check the trace and determinant of A.
3. Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$.
4. Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix}$.
5. Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & 4 \end{bmatrix}$.
6. Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix}$.
7. Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 6 & 3 & 3 \\ 2 & 1 & 1 \\ 8 & 4 & 4 \end{bmatrix}$.
8. Verify Cayley-Hamilton theorem for the matrix $\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$. Hence find the inverse matrix.
9. Verify Cayley-Hamilton theorem for the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{pmatrix}$. Hence find the inverse matrix.
10. Verify Cayley-Hamilton theorem for the matrix $\begin{pmatrix} -1 & 1 & 0 \\ -4 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$. Hence find the inverse matrix.

11. Verify if the matrix is diagonalizable: $A = \begin{bmatrix} 4 & 1 \\ 6 & -1 \end{bmatrix}$.

12. Compute the Diagonal form $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$.

Chapter 6

Vector Spaces

6.1 Introduction

A vector space is a fundamental mathematical structure used in various fields, including physics, engineering, and computer science. It provides a framework for studying linear combinations, transformations, and multi-dimensional spaces.

6.2 Definition and Basic Properties

A vector space V over a field F is a set equipped with two operations:

- Vector addition: $+: V \times V \to V$
- Scalar multiplication: $\cdot: F \times V \to V$

These operations satisfy the following axioms for all $u, v, w \in V$ and all scalars $a, b \in F$:

1. **Associativity of Addition:** $(u + v) + w = u + (v + w)$
1. **Commutativity of Addition:** $u + v = v + u$
2. **Additive Identity:** There exists $0 \in V$ such that $v + 0 = v$ for all $v \in V$.
3. **Additive Inverse:** For each $v \in V$, there exists $-v \in V$ such that $v + (-v) = 0$.
4. **Associativity of Scalar Multiplication:** $a(bv) = (ab)v$.
5. **Distributivity of Scalars over Vector Addition:** $a(u + v) = au + av$.
6. **Distributivity of Scalars over Field Addition:** $(a + b)v = av + bv$.
7. **Multiplicative Identity:** $1v = v$ for all $v \in V$.

Let V be a vector space over a field F, and let $u, v, w \in V$ and $a, b \in F$. The following properties hold:

6.3 Elementary Properties

1. **Uniqueness of the Zero Vector:** There is only one vector 0 in V such that

$$v + 0 = v, \forall v \in V.$$

1. **Uniqueness of Additive Inverses:** For each $v \in V$, there is a unique $-v$ such that

$$v + (-v) = 0.$$

2. **Zero Scalar Multiplication:** For any $v \in V$,

$$0 \cdot v = 0.$$

That is, multiplying any vector by the scalar 0 results in the zero vector.

3. **Zero Vector Scaling:** For any scalar $a \in F$,

$$a \cdot 0 = 0.$$

Scaling the zero vector by any scalar does not change it.

4. **Negation of a Scalar Multiple:** For any $v \in V$ and $a \in F$,

$$(-a)v = -(av).$$

That is, multiplying by a negative scalar is equivalent to negating the vector.

5. **Negation of a Vector is Scalar Multiplication by -1:**

$$(-1)v = -v.$$

6. **Cancellation Law:** If

$$u + v = u + w,$$

then

$$v = w.$$

7. **Scalar Multiplication by Zero Implies Zero Vector:** If $av = 0$ for some nonzero scalar a, then

$$v = 0.$$

This means that if scaling a vector results in the zero vector, the vector must have been zero to begin with.

Examples of Vector Spaces

1. R^n as a Vector Space

The set of all ordered n -tuples of real numbers,

$$R^n = \{(v_1, v_2, \dots, v_n) | v_i \in R\}$$

is a vector space over R, with operations:

$$u + v = (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n)$$

$$cv = (cv_1, cv_2, \dots, cv_n), c \in R$$

Verification:

1. Closure: The sum and scalar multiple of two n -tuples are also n -tuples.
2. Zero vector: $0 = (0,0,\dots,0)$.
3. Additive inverse: $-v = (-v_1, -v_2, \dots, -v_n)$.
4. Commutativity and associativity follow from real number addition.
5. Distributive and associative properties hold due to real number multiplication.

Thus, R^n is a vector space.

2. Polynomial Space $P_n(R)$

The set of all polynomials of degree at most n,

$$P_n(R) = \{a_0 + a_1 x + \cdots + a_n x^n | a_i \in R\}$$

is a vector space over R, with operations:

$$(p + q)(x) = p(x) + q(x), (cp)(x) = c \cdot p(x).$$

Verification:

- Closure: The sum and scalar multiple of polynomials of degree at most n remain polynomials of degree at most n.
- Zero vector: The zero polynomial $0(x) = 0$.
- Additive inverse: $-p(x)$.
- Commutativity, associativity, and distributive properties follow from real number operations.

Thus, $P_n(R)$ is a vector space.

3. Space of Continuous Functions C([a,b])

The set of all continuous functions on $[a, b]$,

$$C([a,b]) = \{f: [a,b] \to R | f \text{is continuous}\}$$

is a vector space over R, with operations:

$$(f + g)(x) = f(x) + g(x), (cf)(x) = c \cdot f(x).$$

Verification:

- Closure: The sum and scalar multiple of continuous functions are continuous.
- Zero vector: $f(x) = 0$.
- Additive inverse: $-f(x)$.

- Commutativity, associativity, and distributive properties hold pointwise.

Thus, $C([a,b])$ is a vector space.

6.4 Vector Subspaces

Definition and Basic Properties

Definition
Let V be a vector space over a field F. A nonempty subset $W \subseteq V$ is called a **vector subspace** of V if it satisfies the following conditions:

1. **Closure under addition:** If $u, v \in W$, then $u + v \in W$.
1. **Closure under scalar multiplication:** If $v \in W$ and $c \in F$, then $cv \in W$.

If a subset W satisfies the two conditions above, then it is automatically a vector space with the same operations as V, and we say that W is a **subspace** of V.

Example 1
The following are always subspaces of any vector space V:

- The **zero subspace** $\{0\}$, consisting of only the zero vector.
- The vector space V itself is a subspace of V.

Example 2
Consider the vector space R^3 with standard vector addition and scalar multiplication. The following are subspaces:

- The set of all vectors of the form $(x, 0,0)$, which forms the x -axis in R^3.
- Any plane through the origin, such as $W = \{(x, y, 0) | x, y \in R\}$.
- The zero subspace $\{(0,0,0)\}$.

The Subspace Criterion

To verify whether a subset is a subspace, we use the following theorem:

Theorem
A nonempty subset W of a vector space V is a subspace of V if and only if for all $u, v \in W$ and $a, b \in F$, the linear combination

$$au + bv \in W.$$

Proof

- Suppose W is a subspace. Then, by definition, it is closed under addition and scalar multiplication. Hence, for any scalars a, b, we have $au \in W$ and $bv \in W$, and their sum is also in W.

- Conversely, if the condition holds for all $u, v \in W$ and scalars a, b, then choosing $a = 1$ and $b = 1$ shows closure under addition, and choosing $b = 0$ shows closure under scalar multiplication.

Examples and Counter examples

Example 3
Consider the set

$$W = \{(x, y, z) \in R^3 | x + y + z = 0\}.$$

To check if W is a subspace:

- Closure under addition: If (x_1, y_1, z_1) and (x_2, y_2, z_2) are in W, then

  $$(x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) = (x_1 + y_1 + z_1) + (x_2 + y_2 + z_2) = 0.$$

  Thus, W is closed under addition.

- Closure under scalar multiplication: If $c \in R$, then

  $$c(x + y + z) = cx + cy + cz = 0.$$

  Thus, W is closed under scalar multiplication.

Since both properties hold, W is a subspace.

Counter Example
Consider the subset

$$S = \{(x, y) \in R^2 | xy = 0\}.$$

It is not a subspace because it is not closed under addition. For example, $(1,0)$ and $(0,1)$ are in S, but their sum $(1,1)$ is not.

Note: Vectors are fundamental objects in linear algebra. They can be "added together" and "scaled" to form new vectors. This chapter introduces "linear combinations", the concept of "spanning" a space, and the key properties of "linear dependence" and "linear independence". These ideas are essential in understanding **vector spaces, basis, and dimension**.

6.5 Linear Combination of Vectors

Definition: Let V be a vector space over a field F, and let $v_1, v_2, \ldots, v_n$ be vectors in V. A vector $w \in V$ is said to be a **linear combination** of $v_1, v_2, \ldots, v_n$ if there exist scalars $c_1, c_2, \ldots, c_n \in F$ such that

$$w = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n.$$

Example 4: Consider the vectors $v_1 = (1,2,3)$ and $v_2 = (2,3,4)$ in R^3. The vector $w = (5,8,11)$ can be written as:

$$w = 3v_1 + v_2.$$

Thus, w is a linear combination of v_1 and v_2.

Remark
The concept of a linear combination is crucial because it helps define the span of a set of vectors, which describes all possible vectors that can be formed from a given set.

6.6 Linear Dependence and Independence and Basis

Definition and Interpretation

Definition: A set of vectors $\{v_1, v_2, \dots, v_n\}$ in a vector space V is called **linearly dependent** if there exist scalars $c_1, c_2, \dots, c_n$, not all zero, such that

$$c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0.$$

This means that at least one of the vectors in the set can be expressed as a linear combination of the others.

Definition: A set of vectors $\{v_1, v_2, \dots, v_n\}$ is said to be **linearly independent** if the only solution to the equation

$$c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$$

is $c_1 = c_2 = \cdots = c_n = 0$.

Examples

Example 5: Consider the vectors:

$$v_1 = (1,2,3), v_2 = (2,4,6), v_3 = (3,6,9)$$

in R^3. These vectors satisfy:

$$2v_1 - v_2 = 0.$$

Since we found nonzero scalars that satisfy the equation, the vectors are linearly dependent.

Example 6: Consider the vectors:

$$v_1 = (1,0,0), v_2 = (0,1,0), v_3 = (0,0,1)$$

in R^3. Suppose:

$$c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1) = (0,0,0).$$

Then, comparing components, we get:

$$c_1 = 0, c_2 = 0, c_3 = 0.$$

Since the only solution is the trivial one, these vectors are linearly independent.

Theorems on Linear Dependence

Theorem 1: Any set of more than n vectors in an n -dimensional vector space is linearly dependent.
Proof
Let V be an n -dimensional vector space, and suppose we have a set of m vectors $\{v_1, v_2, \dots, v_m\}$ with $m > n$.

Since the dimension of V is n, any basis of V consists of exactly n linearly independent vectors. This means that at most n vectors can be linearly independent in V.

Since our set has more than n vectors, at least one of them must be expressible as a linear combination of the others. That is, there exist scalars $c_1, c_2, \dots, c_m$, not all zero, such that:

$$c_1v_1 + c_2v_2 + \cdots + c_mv_m = 0.$$

This shows that the vectors are linearly dependent.
Theorem 2: Any subset of a linearly dependent set is also linearly dependent.
Proof Let $S = \{v_1, v_2, \dots, v_n\}$ be a linearly dependent set. This means there exist scalars $c_1, c_2, \dots, c_n$, not all zero, such that:

$$c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0.$$

Now, consider a subset S' of S. Since S' consists of some or all of the vectors in S, the linear dependence equation above still holds within S'. That is, at least one of the vectors in S' is expressible as a linear combination of the others in S'. Thus, S' is also linearly dependent.

Theorem 3: Any set of vectors in a vector space that contains the zero vector is linearly dependent.

Proof Let $S = \{v_1, v_2, \dots, v_n, 0\}$ be a set of vectors where 0 is the zero vector. Consider the equation:

$$c_1v_1 + c_2v_2 + \cdots + c_nv_n + c_{n+1}0 = 0.$$

Choosing $c_{n+1} \neq 0$ and setting all other coefficients to zero, we get:

$$c_{n+1}0 = 0.$$

Since $c_{n+1} \neq 0$, this provides a nontrivial solution, proving that the set is linearly dependent.

Theorem 4: A set containing two vectors v_1 and v_2 is linearly dependent if and only if one is a scalar multiple of the other.

Proof
Suppose $v_2 = cv_1$ for some scalar c. Then:

$$v_2 - cv_1 = 0.$$

This is a nontrivial linear dependence relation, so $\{v_1, v_2\}$ is linearly dependent.

Now, suppose v_1 and v_2 are linearly dependent. Then there exist scalars c_1, c_2, not both zero, such that:

$$c_1 v_1 + c_2 v_2 = 0.$$

If $c_1 \neq 0$, we can rewrite:

$$v_1 = -\frac{c_2}{c_1} v_2.$$

This shows that one vector is a scalar multiple of the other.

Theorem 5: Any set of three vectors in R^2 is linearly dependent.

Proof The space R^2 has dimension 2, so any basis consists of at most two linearly independent vectors.

If we have three vectors v_1, v_2, v_3 in R^2, then at least one of them must be a linear combination of the others. Thus, there exist scalars c_1, c_2, c_3, not all zero, such that:

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0.$$

This confirms that any three vectors in R^2 are always linearly dependent.

Exercise 6.6.1: Determine whether the following vectors in R^3 are linearly dependent or independent:

$$v_1 = (1,2,3), v_2 = (4,5,6), v_3 = (7,8,9).$$

Exercise 6.6.2: Find a basis for the subspace of R^3 spanned by the vectors

$$(1,2,3), (2,4,6), (3,6,9).$$

Exercise 6.6.3: Prove that if a set of vectors contains the zero vector, it must be linearly dependent.

Exercise 6.6.4: Prove that if $n + 1$ vectors are chosen from an n -dimensional space, they must be linearly dependent.

Exercise 6.65: Determine whether the following vectors in R^3 are linearly dependent or independent:

$$v_1 = (1,2,3), v_2 = (4,5,6), v_3 = (7,8,9).$$

Exercise 6.6.6: Find a basis for the subspace of R^3 spanned by the vectors

$$(1,2,3), (2,4,6), (3,6,9).$$

Exercise 6.6.7: Prove that if a set of vectors contains the zero vector, it must be linearly dependent.

Exercise 6.6.8: Prove that a set of two vectors in R^3 is always linearly dependent if one is a scalar multiple of the other.

Linear Independence and Basis

A set of vectors $\{v_1, v_2, \dots, v_k\}$ is linearly independent if:

$$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = 0 \Rightarrow a_1 = a_2 = \cdots = a_k = 0.$$

Definition A set of vectors $B = \{v_1, v_2, \dots, v_n\}$ in a vector space V is called a **basis** if:

1. B is linearly independent.

1. B spans V, meaning that every vector in V can be written as a linear combination of vectors in B.

The number of elements in any basis is the **dimension** of V.

6.7 Spanning Sets and Basis of a Subspace

Definition
A set of vectors $S = \{v_1, v_2, \dots, v_k\} \subset V$ is said to **span** a subspace W if every vector in W can be written as a linear combination of vectors in S:

$$W = \text{span}(S) = \{c_1 v_1 + c_2 v_2 + \cdots + c_k v_k | c_i \in F\}.$$

Definition
A set $B = \{v_1, v_2, \dots, v_n\}$ is a **basis** for a subspace W if:

1. B spans W.
2. B is linearly independent.

Example 7: Spanning Set in R^2

Consider the set of vectors:

$$S = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}.$$

The set S **spans** R^2 because any vector $v = \begin{bmatrix}x\\y\end{bmatrix}$ can be written as a linear combination:

$$v = a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix} + c\begin{bmatrix}1\\1\end{bmatrix}.$$

However, this set is **not a basis** because it contains three vectors in a 2-dimensional space, meaning it is **linearly dependent**.

Example 8: Basis of R^2

The set:

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

is a **basis** for R^2 because:

2. It is **linearly independent**: If

$$a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0,$$

then $a = 0$ and $b = 0$.

- It **spans** R^2: Any vector in R^2 can be written as a linear combination of these two vectors.

Thus, B is a basis for R^2.

Example 9: Spanning Set and Basis in R^3

Consider the set:

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

This set **spans** R^3 because any vector $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ can be expressed as a linear combination of these four vectors.

However, it is **not a basis** because it contains four vectors in a 3-dimensional space, meaning it is linearly dependent.

A **basis** for R^3 is:

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

This set is linearly independent and spans R^3, so it forms a basis.

Example 10: Basis of the Space of Polynomials P_2

The space of polynomials of degree at most 2 is:

$$P_2 = \{a_0 + a_1 x + a_2 x^2 | a_0, a_1, a_2 \in R\}.$$

A **basis** for P_2 is:

$$B = \{1, x, x^2\}.$$

These three polynomials are **linearly independent**: If

$$a_0(1) + a_1(x) + a_2(x^2) = 0$$

for all x, then $a_0 = a_1 = a_2 = 0$.

They **span** P_2: Any polynomial in P_2 can be written as a linear combination of these three.

Thus, B is a basis of P_2.

Example 11: Infinite-Dimensional Basis – Fourier Series

The space of periodic functions on $[0,2\pi]$, denoted $L^2(0,2\pi)$, has an **infinite basis** given by:

$$B = \{1, \sin x, \cos x, \sin 2x, \cos 2x, \sin 3x, \cos 3x, \ldots\}.$$

This basis is infinite because the space of all periodic functions cannot be spanned by a finite set of functions.

Each vector space has spanning sets and bases, but a basis must be both **spanning and linearly independent**.

Exercises

Exercise 6.7.1
Determine whether the following sets are subspaces of R^3:

1. $W_1 = \{(x, y, z) | x - 2y + 3z = 0\}$.
1. $W_2 = \{(x, y, 1) | x, y \in R\}$.

Exercise 6.7.2
Find a basis for the subspace of R^3 given by

$$W = \text{span}\{(1,2,3), (2,4,6), (3,6,9)\}.$$

Theorem[Replacement Theorem] Let V be a vector space over a field F. Suppose that $S = \{v_1, v_2, \ldots, v_m\}$ is a linearly independent set in V and that $T = \{w_1, w_2, \ldots, w_n\}$ is a spanning set for V. If $m > n$, then S cannot be linearly independent, and if $m \leq n$, then some vectors of T can be replaced by vectors from S to form a new spanning set of V.

Proof Let $S = \{v_1, v_2, \ldots, v_m\}$ be a linearly independent set, and let $T = \{w_1, w_2, \ldots, w_n\}$ be a spanning set of V. Since T spans V, each vector in S can be written as a linear combination of vectors from T.

Step 1: Expressing Vectors of S in Terms of T

Each v_i (for $1 \leq i \leq m$) can be written as:

$$v_i = a_{i1}w_1 + a_{i2}w_2 + \cdots + a_{in}w_n, \text{for some scalars} a_{ij} \in F.$$

This forms a system of m equations in n unknowns.

Step 2: Linear Dependence When $m > n$

If $m > n$, then we have more equations than unknowns. This implies that the system has a nontrivial solution where at least one of the v_i's is a linear combination of the others. This contradicts the assumption that S is linearly independent, proving that S cannot be independent if $m > n$.

Step 3: Replacing Vectors to Form a New Spanning Set

If $m \leq n$, we construct a new spanning set by replacing some vectors in T with vectors from S. We proceed by replacing w_1 with v_1. Since v_1 is a linear combination of the w_j's, we can express w_1 in terms of the remaining vectors and v_1. This replacement preserves the spanning property.

Repeating this process for $v_2, v_3, \ldots, v_m$, we eventually replace m vectors in T, resulting in a new spanning set that includes S. This new set still spans V, as every vector in V can still be written as a linear combination of the updated set.

Thus, we have replaced m vectors from T with the m linearly independent vectors from S, proving the second part of the theorem.

Theorem: *If* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ *are vectors in a vector space* V*, then:*

(a) The set W *of all linear combinations of* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ *is a subspace of* V.

(b) W *is the smallest subspace of* V *that contains* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ *every other subspace of* V *that contains* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ *must contain* W

Proof: *(a)* To show that W is a subspace of V, it must be proven that it is closed under addition and scalar multiplication. There is at least one vector in W, namely, $\mathbf{0}$, since $\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_r$. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W, then

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_r\mathbf{v}_r$$

and

$$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r$$

where $c_1, c_2, \ldots, c_r, k_1, k_2, \ldots, k_r$ are scalars. Therefore

$$\mathbf{u} + \mathbf{v} = (c_1 + k_1)\mathbf{v}_1 + (c_2 + k_2)\mathbf{v}_2 + \cdots + (c_r + k_r)\mathbf{v}_r$$

and, for any scalar k, $k\mathbf{u} = (kc_1)\mathbf{v_1} + (kc_2)\mathbf{v_2} + \cdots + (kc_r)\mathbf{v_r}$.

Thus, u + v and ku are linear combinations of $v_1, v_2, \ldots, v_r$ and consequently lie in W. Therefore, W is closed under addition and scalar multiplication.

(b) Each vector $\mathbf{v_i}$ is a linear combination of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ since we can write

$$\boldsymbol{v_i} = 0\boldsymbol{v_1} + 0\boldsymbol{v_2} + \cdots + 1\boldsymbol{v_i} + \cdots + 0\boldsymbol{v_r}$$

Therefore, the subspace W contains each of the vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$. Let W' be any other subspace that contains $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$. Since W' is closed under addition and scalar multiplication, it must contain all linear combinations of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$. Thus W' contains each vector of W.

Theorem: Let S = {v1, v2, . . . , vr} and S' = {w1, w2, . . . , wk} be two sets of vectors in a vector space V . Then span(S) = span(S') if and only if each vector in S is a linear combination of those in S' and (conversely) each vector in S' is a linear combination of those in S.

Proof. If each vector in S is a linear combination of those in S' then span(S) ⊆ span(S') and if each vector in S' is a linear combination of those in S then span(S') ⊆ span(S) and therefore span(S) = span(S').

If $$v_i = a_1w_1 + a_2w_2 + \cdots + a_nw_n$$

for all possible $a_1, a_2, \ldots, a_n$ then $v_i \in$ span(S) but vi /∈ span(S') therefore and vice versa.

6.8 Dimension of a Vector Space

Definition: The dimension of a vector space V, denoted $dim(V)$, is the number of vectors in any basis of V.

Example: The dimension of R^n is n, since the standard basis consists of n vectors $\{(1,0,\ldots,0),(0,1,\ldots,0),\ldots,(0,0,\ldots,1)\}$.

Theorem: Any two bases of a vector space have the same number of elements.

Proof Let B_1 and B_2 be two bases of V. Assume B_1 has more elements than B_2. Then, since B_1 spans V, every vector in B_1 can be expressed as a linear combination of vectors in B_2, contradicting linear independence of B_1. Similarly, reversing the roles of B_1 and B_2, we obtain a contradiction, proving that both bases must have the same number of elements.

Exercises

Exercise 1: Determine whether the set $\{(1,0,0), (0,1,0), (0,0,1)\}$ forms a basis for R^3.

Exercise 2: Find the dimension of the space of all polynomials of degree at most 3.

Exercise 3: Prove that if V is a vector space of dimension n, any set of n linearly independent vectors is a basis of V.

Existence of a Basis (Spanning Set and Linear Independence)

Theorem: Every vector space has a basis.

Proof: Let V be a vector space. If $V = \{0\}$, the trivial space, then the set $\{0\}$ is a basis. For non-trivial vector spaces, take any spanning set of V, say $S = \{v_1, v_2, ...\}$. If S is linearly independent, it is a basis. If not, remove vectors from S until you obtain a linearly independent set. This set must span V by the definition of a spanning set. Thus, V has a basis.

2. Uniqueness of Basis (Dimension Theorem)

Theorem: Any two bases of a vector space V have the same number of elements.

Proof: Let $B_1 = \{b_1, b_2, ..., b_n\}$ and $B_2 = \{c_1, c_2, ..., c_m\}$ be two bases of V. Assume for contradiction that $n \neq m$. Without loss of generality, assume $n < m$. Then $\{b_1, b_2, ..., b_n\}$ is a linearly independent set of vectors. Since B_2 is a basis, each b_i for $1 \leq i \leq n$ can be written as a linear combination of the vectors in B_2. This contradicts the fact that B_2 is linearly independent. Therefore, $n = m$, so all bases of V have the same size.

3. Dimension of a Subspace

Theorem: If W is a subspace of a vector space V, then the dimension of W is less than or equal to the dimension of V.

Proof: Let $B_W = \{w_1, w_2, ..., w_k\}$ be a basis of W, and $B_V = \{v_1, v_2, ..., v_m\}$ be a basis of V. Since B_W is a set of linearly independent vectors in V, and B_V is a spanning set for V, we have $k \leq m$, so $dim(W) \leq dim(V)$.

4. Extension of a Linearly Independent Set

Theorem: If $S = \{v_1, v_2, ..., v_n\}$ is a linearly independent set of vectors in a vector space V, then it can be extended to a basis of V.

Proof: Let S be a linearly independent set in V. If S spans V, then S is already a basis. Otherwise, take any vector $v \in V$ that is not in the span of S. Add this vector to S, and continue adding vectors from V that are not in the span of the set so far, maintaining linear independence. This process must eventually result in a basis of V.

5. Rank-Nullity Theorem

Theorem: For a linear transformation $T: V \to W$ between vector spaces V and W, we have

$$dim\big(\ker(T)\big) + dim\big(\mathrm{im}(T)\big) = dim(V)$$

Proof: Let $\{v_1, v_2, \dots, v_k\}$ be a basis for the kernel of T, and extend it to a basis $\{v_1, v_2, \dots, v_k, v_{k+1}, \dots, v_n\}$ of V. The set $\{T(v_{k+1}), T(v_{k+2}), \dots, T(v_n)\}$ is linearly independent and spans the image of T, so $dim\big(\mathrm{im}(T)\big) = n - k$. Therefore,

$$dim\big(\ker(T)\big) + dim\big(\mathrm{im}(T)\big) = k + (n - k) = n = dim(V)$$

Thus, the rank-nullity theorem holds.

6. Independence of the Columns of a Matrix

Theorem: The columns of an $m \times n$ matrix are linearly independent if and only if the rank of the matrix is equal to n.

Proof: Let A be an $m \times n$ matrix. The columns of A are linearly independent if the only solution to $Ax = 0$ is $x = 0$. This implies that the nullity of A is 0. By the rank-nullity theorem, we know that the rank of A (the number of linearly independent columns) is n. Therefore, the columns are linearly independent if and only if the rank of A is n. ▯

6.9 Linear Transformations

A function $T: V \to W$ between vector spaces is a linear transformation if:

$$T(av + bw) = aT(v) + bT(w) \text{for all} v, w \in V \text{and} a, b \in F.$$

6.10 Inner Product Spaces

An inner product on a vector space V is a function $\langle \cdot,\cdot \rangle: V \times V \to F$ satisfying:

1. $\langle v, v \rangle \geq 0$, with equality if and only if $v = 0$.
1. $\langle u, v \rangle = \overline{\langle v, u \rangle}$.
2. $\langle au + bv, w \rangle = a\langle u, w \rangle + b\langle v, w \rangle$.

6.11 Applications of Vector Spaces

Vector spaces have wide-ranging applications in:

- Quantum mechanics (Hilbert spaces)
- Engineering (Signal processing, Control systems)
- Machine learning (Feature vector spaces)

- Computer graphics (Transformations and projections)

6.12 Illustrative Examples

1. Determine whether the set S={(1,2,3),(4,5,6)} is a basis for R^3.

Solution:

To check if $S = \{(1,2,3),(4,5,6)\}$ is a basis for R^3, we need to check two things:

- Linear independence.
- Spanning R^3.

Step 1: Linear independence check.

The set S contains only two vectors, so we cannot form a basis for R^3, since R^3 has dimension 3. A basis for R^3 must have three linearly independent vectors. However, we will still check if these two vectors are linearly independent.

To check for linear independence, we need to solve the equation:

$$c_1(1,2,3) + c_2(4,5,6) = (0,0,0)$$

This gives the system of equations:

$$c_1 + 4c_2 = 0$$

$$2c_1 + 5c_2 = 0$$

$$3c_1 + 6c_2 = 0$$

Solving the first equation for c_1, we get $c_1 = -4c_2$. Substituting this into the second equation:

$$2(-4c_2) + 5c_2 = 0 \Longrightarrow -8c_2 + 5c_2 = 0 \Longrightarrow -3c_2 = 0 \Longrightarrow c_2 = 0$$

Therefore, $c_2 = 0$ and $c_1 = 0$.

Since the only solution is $c_1 = c_2 = 0$, the vectors are linearly independent.

Step 2: Conclusion.

Since the set S contains only two vectors and we are working in R^3, the set cannot span R^3 because it does not have enough vectors to span a 3-dimensional space.

Therefore, S is not a basis for R^3.

2. Prove that $\boldsymbol{S = \{(1,0,0),(0,1,0),(0,0,1)\}}$ is a basis for R^3.

Solution:

To prove that $S = \{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis for R^3, we need to verify that:

- The set S is linearly independent.
- The set S spans R^3.

Step 1: Linear independence.

We check if the set S is linearly independent by solving the equation:

$$c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1) = (0,0,0)$$

This leads to the system of equations:

$$c_1 = 0$$

$$c_2 = 0$$

$$c_3 = 0$$

The only solution is $c_1 = c_2 = c_3 = 0$, which shows that the set S is linearly independent.

Step 2: Spanning.

To show that S spans R^3, we need to show that any vector $(x, y, z) \in R^3$ can be written as a linear combination of the vectors in S. Let:

$$(x, y, z) = c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1)$$

This gives the system of equations:

$$c_1 = x$$

$$c_2 = y$$

$$c_3 = z$$

Therefore, any vector in R^3 can be written as a linear combination of $(1,0,0), (0,1,0), (0,0,1)$, meaning S spans R^3.

Conclusion:

Since S is both linearly independent and spans R^3, S is a basis for R^3.

3. Find a basis for the subspace W={(x,y,z)ϵR^3, $x - y + z = 0$}

Solution:

The subspace W is defined by the equation $x - y + z = 0$. We will find a basis for this subspace and determine its dimension.

Step 1: Express the equation in terms of free variables.

From $x - y + z = 0$, we can solve for x:

$$x = y - z$$

Therefore, every vector in W can be written as:

$$(x, y, z) = (y - z, y, z) = y(1,1,0) + z(-1,0,1)$$

Step 2: Find linearly independent vectors.

The vectors $(1,1,0)$ and $(-1,0,1)$ are linearly independent because neither is a scalar multiple of the other. Thus, these two vectors form a basis for W.

Step 3: Determine the dimension of W.

Since the basis for W consists of two vectors, the dimension of W is 2.

Conclusion:

A basis for W is $\{(1,1,0), (-1,0,1)\}$. The dimension of W is 2.

4. Prove that the set of all 2x2 matrices with real entries $M_2(R)$ forms a vector space.

Solution:

To prove that $M_2(R)$, the set of all 2x2 matrices with real entries, is a vector space, we must verify that it satisfies all the axioms of a vector space.

Step 1: Closure under addition.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ be two matrices in $M_2(R)$. Their sum is:

$$A + B = \begin{pmatrix} a + e & b + f \\ c + g & d + ? \end{pmatrix}$$

Since the sum of two real numbers is a real number, the resulting matrix is also a 2x2 matrix with real entries. Thus, $A + B \in M_2(R)$.

Step 2: Closure under scalar multiplication.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix in $M_2(R)$, and let $r \in R$ be a scalar. The scalar multiple is:

$$rA = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix}$$

Since the product of a real number and a real number is a real number, the resulting matrix is a 2x2 matrix with real entries. Thus, $rA \in M_2(R)$.

Step 3: Verify other axioms.

The other axioms (commutativity and associativity of addition, existence of additive identity and inverses, distributivity of scalar multiplication, and multiplicative identity of scalar multiplication) can be easily verified using properties of real numbers and matrix operations.

Conclusion:

Since $M_2(R)$ satisfies all the axioms of a vector space, we conclude that the set of all 2x2 matrices with real entries forms a vector space.

5. Determine whether the set $\boldsymbol{S = \{(1,2,3),(4,5,6)\}}$ is a basis for $\boldsymbol{R^3}$.

Solution:

To check if $S = \{(1,2,3),(4,5,6)\}$ is a basis for R^3, we need to check two things:

- Linear independence.
- Spanning R^3.

Step 1: Linear independence check.

The set S contains only two vectors, so we cannot form a basis for R^3, since R^3 has dimension 3. A basis for R^3 must have three linearly independent vectors. However, we will still check if these two vectors are linearly independent.

To check for linear independence, we need to solve the equation:

$$c_1(1,2,3) + c_2(4,5,6) = (0,0,0)$$

This gives the system of equations:

$$c_1 + 4c_2 = 0$$

$$2c_1 + 5c_2 = 0$$

$$3c_1 + 6c_2 = 0$$

Solving the first equation for c_1, we get $c_1 = -4c_2$. Substituting this into the second equation:

$$2(-4c_2) + 5c_2 = 0 \Longrightarrow -8c_2 + 5c_2 = 0 \Longrightarrow -3c_2 = 0 \Longrightarrow c_2 = 0$$

Therefore, $c_2 = 0$ and $c_1 = 0$.

Since the only solution is $c_1 = c_2 = 0$, the vectors are linearly independent.

Step 2: Conclusion.

Since the set S contains only two vectors and we are working in R^3, the set cannot span R^3 because it does not have enough vectors to span a 3-dimensional space.

Therefore, S is not a basis for R^3.

6. Prove that $\boldsymbol{S = \{(1,0,0),(0,1,0),(0,0,1)\}}$ is a basis for $\boldsymbol{R^3}$.

Solution:

To prove that $S = \{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis for R^3, we need to verify that:

- The set S is linearly independent.
- The set S spans R^3.

Step 1: Linear independence.

We check if the set S is linearly independent by solving the equation:

$$c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1) = (0,0,0)$$

This leads to the system of equations:

$$c_1 = 0$$

$$c_2 = 0$$

$$c_3 = 0$$

The only solution is $c_1 = c_2 = c_3 = 0$, which shows that the set S is linearly independent.

Step 2: Spanning.

To show that S spans R^3, we need to show that any vector $(x, y, z) \in R^3$ can be written as a linear combination of the vectors in S. Let:

$$(x, y, z) = c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1)$$

This gives the system of equations:

$$c_1 = x$$

$$c_2 = y$$

$$c_3 = z$$

Therefore, any vector in R^3 can be written as a linear combination of $(1,0,0), (0,1,0), (0,0,1)$, meaning S spans R^3.

Conclusion:

Since S is both linearly independent and spans R^3, S is a basis for R^3.

7. Find a basis for the subspace $\boldsymbol{W = \{(x, y, z) \in R^3 : x - y + z = 0\}}$.

Solution:

The subspace W is defined by the equation $x - y + z = 0$. We will find a basis for this subspace and determine its dimension.

Step 1: Express the equation in terms of free variables.

From $x - y + z = 0$, we can solve for x:

$$x = y - z$$

Therefore, every vector in W can be written as:

$$(x, y, z) = (y - z, y, z) = y(1,1,0) + z(-1,0,1)$$

Step 2: Find linearly independent vectors.

The vectors $(1,1,0)$ and $(-1,0,1)$ are linearly independent because neither is a scalar multiple of the other. Thus, these two vectors form a basis for W.

Step 3: Determine the dimension of W.

Since the basis for W consists of two vectors, the dimension of W is 2.

Conclusion:

A basis for W is $\{(1,1,0), (-1,0,1)\}$. The dimension of W is 2.

8. Prove that the set of all 2x2 matrices with real entries $M_2(R)$ forms a vector space.

Solution:

To prove that $M_2(R)$, the set of all 2x2 matrices with real entries, is a vector space, we must verify that it satisfies all the axioms of a vector space.

Step 1: Closure under addition.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ be two matrices in $M_2(R)$. Their sum is:

$$A + B = \begin{pmatrix} a + e & b + f \\ c + g & d + \mathrm{h} \end{pmatrix}$$

Since the sum of two real numbers is a real number, the resulting matrix is also a 2x2 matrix with real entries. Thus, $A + B \in M_2(R)$.

Step 2: Closure under scalar multiplication.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix in $M_2(R)$, and let $r \in R$ be a scalar. The scalar multiple is:

$$rA = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix}$$

Since the product of a real number and a real number is a real number, the resulting matrix is a 2x2 matrix with real entries. Thus, $rA \in M_2(R)$.

Step 3: Verify other axioms.

The other axioms (commutativity and associativity of addition, existence of additive identity and inverses, distributivity of scalar multiplication, and multiplicative identity of scalar multiplication) can be easily verified using properties of real numbers and matrix operations.

Conclusion:

Since $M_2(R)$ satisfies all the axioms of a vector space, we conclude that the set of all 2x2 matrices with real entries forms a vector space.

9. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$. Find $A + B$ and $2A - B$.

Solution:

First, compute $A + B$:

$$A + B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$$

Now, compute $2A - B$:

$$2A = 2 \times \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$$

$$2A - B = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix} - \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 2-5 & 4-6 \\ 6-7 & 8-8 \end{pmatrix} = \begin{pmatrix} -3 & -2 \\ -1 & 0 \end{pmatrix}$$

Conclusion:

The sum $A + B = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$ and the difference $2A - B = \begin{pmatrix} -3 & -2 \\ -1 & 0 \end{pmatrix}$.

10. Prove that if a set S is linearly independent, then every subset of S is linearly independent.

Solution:

Let $S = \{v_1, v_2, \dots, v_n\}$ be a linearly independent set. We need to prove that every subset of S is linearly independent.

Suppose $T = \{v_1, v_2, \dots, v_k\}$ is a subset of S. We want to prove that T is linearly independent. By definition, T is linearly independent if the only solution to the equation:

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$$

is $c_1 = c_2 = \cdots = c_k = 0$.

Since S is linearly independent, the equation

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$$

has only the trivial solution $c_1 = c_2 = \cdots = c_n = 0$.

Since T is a subset of S, the equation $c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$ is a restriction of the linear combination equation for S. Therefore, the only solution to this equation is also $c_1 = c_2 = \cdots = c_k = 0$, which shows that T is linearly independent.

Conclusion:

If a set S is linearly independent, then every subset of S is linear

11. Prove that the set of all $n \times n$ invertible matrices does not form a vector space.

Solution: A vector space must be closed under addition and scalar multiplication.

Consider two invertible matrices A and B. Their sum $A + B$ may not be invertible. For example:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Then,

$$A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

which is not invertible. Hence, the set of invertible matrices is **not closed under addition** and does not form a vector space.

12. Prove that if U and W are subspaces of V, then $U \cap W$ is also a subspace of V.

Solution: We verify the subspace conditions:

- **Contains the zero vector:** Since U, W are subspaces, they contain 0. Thus, $0 \in U \cap W$.
- **Closed under addition:** If $u, w \in U \cap W$, then $u, w \in U$ and $u, w \in W$. Since U and W are subspaces, $u + w \in U$ and $u + w \in W$, so $u + w \in U \cap W$.
- **Closed under scalar multiplication:** If $v \in U \cap W$ and $c \in F$, then $cv \in U$ and $cv \in W$. Thus, $cv \in U \cap W$.

Since all conditions hold, $U \cap W$ is a subspace of V.

13. Find a basis and dimension of the space of all upper triangular 3×3 matrices.

Solution: An upper triangular 3×3 matrix is of the form:

$$A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}.$$

There are six independent parameters a, b, c, d, e, f, so the space has **dimension 6**.

A basis consists of matrices with a single nonzero entry in each independent position:

$$B = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

Thus, the **dimension is 6**.

14. Prove that every finite-dimensional vector space is isomorphic to F^n for some n.

Solution: Let V be a finite-dimensional vector space with basis $B = \{v_1, v_2, \dots, v_n\}$. Define a map:

$$\phi: V \to F^n, v = c_1v_1 + c_2v_2 + \cdots + c_nv_n \mapsto (c_1, c_2, \dots, c_n).$$

- **Injectivity:** If $\phi(v) = 0$, then $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$, implying $c_1 = c_2 = \cdots = c_n = 0$, so $v = 0$. Thus, ϕ is injective. - **Surjectivity:** Any $(c_1, c_2, \dots, c_n) \in F^n$ corresponds to $c_1v_1 + c_2v_2 + \cdots + c_nv_n \in V$, so ϕ is surjective.

Since ϕ is a **bijective linear map**, $V \cong F^n$.

15. Show that the row space of a matrix is equal to the column space of its transpose.

Solution: Let A be an $m \times n$ matrix. Its **row space** is the subspace spanned by its row vectors. The **column space** of A^T consists of the same vectors as the row space of A, since transposing a matrix swaps rows and columns. Thus,

$$\text{Row space of} A = \text{Column space of} A^T.$$

16. Prove that if $dimV = n$, then any generating set of V with n elements is a basis.

Solution: Let $S = \{v_1, v_2, \dots, v_n\}$ be a spanning set of V. - If S were linearly dependent, we could remove an element without losing the spanning property, contradicting that V has dimension n. - Hence, S must be **linearly independent** and is therefore a basis.

Exercise 6

1. Which of the following is NOT a requirement for a set to be a vector space?

   (a) Closure under vector addition

   (b) Closure under scalar multiplication

   (c) The presence of a multiplicative inverse for every vector

   (d) The existence of a zero vector

   Answer: (c)

2. If a set V is a vector space, then which of the following is always true?

   (a) V contains exactly one zero vector

   (b) V contains at least one zero vector

   (c) V does not contain a zero vector

(d) V contains infinitely many zero vectors

Answer: (a)

3. A subset W of a vector space V is a subspace if:

(a) W is closed under addition and scalar multiplication

(b) W is non-empty

(c) W contains only the zero vector

(d) W is finite

Answer: (a)

4. The set of all solutions to the equation $ax + by = 0$ in R^2 forms:

(a) A vector space

(b) A subspace of R^2

(c) A basis of R^2

(d) Not a vector space

Answer: (b)

5. A basis of a vector space is:

(a) A set of linearly dependent vectors

(b) A set of vectors that spans the space

(c) A maximal set of linearly dependent vectors

(d) A set that contains only the zero vector

Answer: (b)

6. The dimension of a vector space is:

(a) The number of vectors in the spanning set

(b) The number of vectors in any basis

(c) The number of linearly dependent vectors in the space

(d) The number of vectors in the largest basis

Answer: (b)

7. If a set of vectors spans a vector space, then:

(a) The set must be linearly independent

(b) The set must be a basis

(c) The set must be finite

(d) Every vector in the space can be written as a linear combination of the set

Answer: (d)

8. The rank of a matrix is:

(a) The number of rows in the matrix

(b) The number of nonzero rows in its row echelon form

(c) The number of pivot columns in its row echelon form

(d) The number of zero rows in the matrix

Answer: (c)

9. A set of vectors is linearly dependent if:

(a) At least one vector can be written as a linear combination of the others

(b) The determinant of the matrix formed by these vectors is nonzero

(c) All the vectors in the set are nonzero

(d) The vectors span the entire space

Answer: (a)

10. The trivial solution to a homogeneous system of linear equations is:

(a) The zero solution

(b) Any nonzero solution

(c) The determinant of the coefficient matrix

(d) The dimension of the null space

Answer: (a)

11. A vector space with a finite basis is called:

(a) Infinite-dimensional

(b) One-dimensional

(c) Finite-dimensional

(d) Unbounded

Answer: (c)

12. If the number of vectors in a set is greater than the dimension of the space, then the set is:

(a) Linearly independent

(b) Linearly dependent

(c) A basis

(d) Empty

Answer: (b)

13. The zero vector in a vector space is unique because:

(a) There can be multiple zero vectors

(b) It satisfies the axioms of a vector space

(c) The zero vector depends on the basis

(d) The definition of a vector space requires exactly one zero vector

Answer: (d)

14. The standard basis for R^3 consists of:

(a) Any three linearly independent vectors

(b) Three mutually orthogonal unit vectors

(c) Any three vectors that span R^3

(d) The zero vector

Answer: (b)

15. A subspace of a vector space must:

(a) Contain the zero vector

(b) Be finite

(c) Contain at least two linearly independent vectors

(d) Contain only nonzero vectors

Answer: (a)

16. The column space of a matrix is:

    (a) The space spanned by its row vectors

    (b) The space spanned by its column vectors

    (c) The space spanned by its eigenvectors

    (d) The set of all possible linear transformations of the matrix

Answer: (b)

17. The rank-nullity theorem states that:

    (a) $\text{rank}(A) + \text{nullity}(A) = \text{number of columns of} A$

    (b) $\text{rank}(A) + \text{nullity}(A) = \text{number of rows of} A$

    (c) $\text{rank}(A) = \text{nullity}(A)$

    (d) The rank is always equal to the nullity

Answer: (a)

Short Answer Questions

1. Define a vector space with an example.
2. What is a subspace? Give an example.
3. State the conditions for a subset of a vector space to be a subspace.
4. Define linear dependence and linear independence of vectors.
5. What is the dimension of a vector space? How is it determined?
6. Give an example of a vector space of dimension 3.
7. What is the zero vector in a vector space, and why is it unique?
8. If a set of vectors spans a vector space, what does it mean?
9. Define basis of a vector space with an example.
10. State and explain the rank-nullity theorem in brief.

Long Answer Questions

1. Prove that the intersection of two subspaces of a vector space is also a subspace.
2. Show that the set of all polynomials of degree at most n forms a vector space.
3. Prove that a set of vectors in a vector space is linearly dependent if and only if at least one vector in the set can be expressed as a linear combination of the others.
4. Find a basis and the dimension of the solution space of the system:

$$x + 2y + 3z = 0, 2x + 3y + 4z = 0.$$

5. Prove that the union of two subspaces is not necessarily a subspace.
6. Explain with proof: Any finite-dimensional vector space has a basis.
7. Find the dimension and a basis for the null space of the matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

8. Prove that any basis of a finite-dimensional vector space has the same number of elements.
9. Let $\{v_1, v_2, v_3\}$ be a linearly dependent set of vectors. Show that at least one of them can be written as a linear combination of the others.
10. If $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis of a vector space V, prove that every vector in V can be uniquely expressed as a linear combination of the vectors in β.

Chapter 7

Inner Product Spaces and Orthogonality

7.1. Introduction

An orthogonal vector space is an inner product space where the concept of orthogonality (perpendicularity) between vectors is well-defined. This concept plays a crucial role in various domains, including linear algebra, computer graphics, machine learning, and signal processing.

Orthogonality is a fundamental property that simplifies computations, facilitates geometric interpretations, and underpins various mathematical and engineering techniques. The study of orthogonal vector spaces enables efficient transformations, decompositions, and optimizations in numerous applications.

7.2 Dot product of R^n

The **inner product** or **dot product** of R^n is a function $\langle , \rangle$ defined by

$\langle \mathbf{u}, \mathbf{v} \rangle = a_1b_1 + a_2b_2 + \cdots + a_nb_n$ for $u = [a_1, a_2, . . . , a_n]^T$, $\boldsymbol{v} = [b_1, b_2, . . . , b_n]^T \in R^n$.

The inner product $\langle , \rangle$ satisfies the following properties:

(1) **Linearity**: $\langle a\boldsymbol{u} + b\boldsymbol{v}, \boldsymbol{w} \rangle = a\langle \boldsymbol{u}, \boldsymbol{w} \rangle + b\langle \boldsymbol{v}, \boldsymbol{w} \rangle$.

(2) **Symmetric Property**: $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle$.

(3) **Positive Definite Property**: For any $\boldsymbol{u} \in V$, $\langle \boldsymbol{u}, \boldsymbol{u} \rangle \geq 0$; and $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0$ if and only if $\boldsymbol{u} = 0$.

With the dot product we have geometric concepts such as the length of a vector, the angle between two vectors, orthogonality, etc. We shall push these concepts to abstract vector spaces so that geometric concepts can be applied to describe abstract vectors.

7.3. Inner product spaces

Definition 2.1. An **inner product** of a real vector space V is an assignment that for any two vectors
$u, v \in V$, there is a real number $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$, satisfying the following properties:

(4) **Linearity**: $\langle a\boldsymbol{u} + b\boldsymbol{v}, \boldsymbol{w} \rangle = a\langle \boldsymbol{u}, \boldsymbol{w} \rangle + b\langle \boldsymbol{v}, \boldsymbol{w} \rangle$.

(5) **Symmetric Property**: $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle$.

(6) **Positive Definite Property**: For any $\boldsymbol{u} \in V$, $\langle \boldsymbol{u}, \boldsymbol{u} \rangle \geq 0$; and $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0$ if and only if $\boldsymbol{u} =$

0. The vector space V with an inner product is called a (**real**) **inner product space**.
S

Example 7.3.1. For $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$, define

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 2x1y1 - x1y2 - x2y1 + 5x2y2.$$

Then, $\langle \cdot,\cdot \rangle$ is an inner product on $\mathbb{R}^2$. It is easy to see the linearity and the symmetric property. As for the positive definite property, note that

$$< x_1, x_2 > = 2x_1^2 - 2x_1x_2 + 5x_2^2$$
$$= (x_1 + x_2)^2 + (x_1 - 2x_2)^2 \geq 0.$$

Moreover, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ if and only if

$$x + x_2 = 0, \quad x_1 - 2x_2 = 0,$$

which implies $x_1 = x_2 = 0$, i.e., $\boldsymbol{x} = \boldsymbol{0}$. This inner product on $\mathbb{R}^2$ is different from the dot product of $\mathbb{R}^2$.

For each vector $u \in V$, the norm (also called the length) of u is defined as the number

$$||u|| := \sqrt{< u, u >}.$$

If $\|\boldsymbol{u}\| = 1$, we call $\boldsymbol{u}$ a **unit vector** and $\boldsymbol{u}$ is said to be **normalized**. For any nonzero vector $\boldsymbol{v} \in V$, we have the unit vector

$$\hat{v} := \frac{v}{||v||}.$$

This process is called normalizing *v*.

Let $\boldsymbol{B} = \boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ be a basis of an *n*-dimensional inner product space **V**. For vectors $u, v \in V$, write

$$u = x_1u_1 + x_2u_2 + \cdots + x_nu_n;$$
$$v = y_1u_1 + y_2u_2 + \cdots + y_nu_n$$

The linearity implies

$$< u, v > = < \sum_{i=1}^{n} x_iu_i, \sum_{j=1}^{n} y_ju_j >$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} x_iy_i < u_i, u_j >.$$

We call the $n \times n$ matrix

$$A = \begin{pmatrix} < u_1, u_1 > & \cdots & < u_1, u_n > \\ \vdots & \ddots & \vdots \\ < u_n, u_1 > & \cdots & < u_n, u_n > \end{pmatrix}$$

the matrix of the inner product <,> relative to the basis $\boldsymbol{B}$. Thus, using coordinate vectors

$[u]_B = [x_1, x_2, \ldots, x_n]^T, [v]_B = [y_1, y_2, \ldots y_n]^T$, we have

$$< u, v > = [u]_B^T A [v]_B$$

Examples of inner product spaces

Example 7.3.2. The vector space R^n with the dot product

$$< \boldsymbol{u} \cdot \boldsymbol{v} >= a_1b_1 + a_2b_2 + \cdots + a_nb_n,$$

where $\boldsymbol{u} = [a_1, a_2, \ldots, a_n]^T$, $\boldsymbol{v} = [b_1, b_2, \ldots, b_n]^T \in \mathbb{R}^n$, is an inner product space. The vector space R^n with this special inner product (dot product) is called the **Euclidean *n*-space**, and the dot product is called the **standard inner product** on $\mathbb{R}^n$.

Example 7.3.3. The vector space C[a;b] of all real-valued continuous functions on a closed interval [a; b] is an inner product space, whose inner product is defined by

$$< f, g >= \int_a^b f(t)g(t)dt \quad f, g \in C[a, b].$$

Example 7.3.4. The vector space $M_{m,n}$ of all $m \times n$ real matrices can be made into an inner product space under the inner product

$< A, B >= tr(B^T A)$, where $A, B \in M_{m,n}$.

7.4 Representation of inner product

Theorem 4.1. Let V be an n-dimensional vector space with an inner product <,>, and let A be the matrix of $<,>$ relative to a basis B. Then for any vectors $u;\ v \in V$,

$$< u, v > = x^T A y$$

where x and y are the coordinate vectors of u and v, respectively, i.e., $x = [u]_B$ and $y = [v]_B$.

Example 4.1. For the inner product of R3 defined by

$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 2x_1y_1 - x_1y_2 - x_2y_1 + 5x_2y_2,$

where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$, its matrix relative to the standard basis $E = \{e_1, e_2\}$ is

$A = \begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix}$.

The inner product can be written as

$< x, y > = x^T A y = (x_1, x_2) \begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

Theorem 4.2. Let V be a ¯nite-dimensional inner product space. Let A;B be matrices of the inner product relative to bases B, B' of V , respectively. If P is the transition matrix from B to B′. Then $B = P^T A P$.

7.5 Cauchy-Schwarz inequality

Theorem 7.5.1 (Cauchy-Schwarz Inequality). For any vectors u; v in an inner product space V,

$$|< u, v >| \leq \|u\| \|v\|.$$

Proof: Consider the function

$$y = y(t) := < u + tv; u + tv >; \ t \in \mathbb{R}$$

Then $y(t)$ ≥0 by the third property of inner product. Note that y(t) is a quadratic function of t. In fact,

$$y(t) = < u, u + tv > + < tv, u + tv >$$
$$= < u; u > + 2 < u, v > t + < v, v > t^2$$

Thus, the quadratic equation

$$< u; u > + 2 < u, v > t + < v, v > t^2 = 0$$

has at most one solution as y(t) ≥ 0. This implies that its discriminant must be less or equal to zero, i.e.,

$(2 < u, v >)^2 - 4 < u, u >< v, v > \leq 0.$

The Cauchy-Schwarz inequality follows.

Theorem 7.5.2. The norm in an inner product space V satisfies the following properties:

(N1) $\|u\| \geq 0$ *; and* $\|u\|$ = 0 *if and only if* u = 0.

(N2) $\|cu\| = c\ \|u\|$.

(N3) $\|u + v\| \leq \|u\| + \|v\|$

For nonzero vectors *u; v 2 V* , the Cauchy-Schwarz inequality implies

$$-1 \leq \frac{< u, v >}{\|u\|\ \|v\|} \leq 1$$

angle μ between *u* and *v* is defined by

$$\cos\theta = \frac{< u, v >}{\|u\|\ \|v\|}$$

The angle exists and is unique.

7.6 Orthogonality

Let V be an inner product space. Two vectors $u, v \in V$ are said to be orthogonal if

$< u, v >= 0.$

Example 7.6.1. For inner product space $C[-\pi, \pi]$, the functions sin *t* and cos *t* are orthogonal as

$$< \sin t, \cos t \geq \int_{-\pi}^{\pi} \sin t\ \cos t\ dt$$
$$= \frac{1}{2}\sin^2 t\ |_{-\pi}^{\pi} = 0$$

Example 7.6.2. Let $u = [a_1;\ a_2; ::: ;\ a_n]^T \in \mathbb{R}^n$. The set of all vector of the Euclidean *n*-space $\mathbb{R}^n$ that are orthogonal to *u* is a subspace of $\mathbb{R}^n$. In fact, it is the solution space of the single linear equation

$$< u, x >= a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$

Let S be a nonempty subset of an inner product space V. We denote by $S^{\perp}$ the set of all vectors of V that are orthogonal to every vector of S, called the orthogonal complement of S in V. In notation, $S^{\perp} = \{u \in V: < u, v > = 0 \ for \ every \ v \in S\}$.

If S contains only one vector u, we write

$u^{\perp} = \{v \in V: < u, v > = 0 \}$.

Proposition 7.6.1. Let S be a nonempty subset of an inner product space V. Then the orthogonal complement $S^{\perp}$ is a subspace of V.

Proof: To show that $S^{\perp}$ is a subspace. We need to show that $S^{\perp}$ is closed under addition and scalar multiplication. Let $u, v \in S^{\perp}$ and $c \in \mathbb{R}$. Since $: < u, w > = 0$ and $: < v, w > = 0$ for all $w \in S$ then

$$< u + v, w > = < u, w > + < v, w > = 0$$

$$< cu, w > = c < u, w > = 0$$

for all $w \in S$. So $u + v, cu \in S^{\perp}$. Hence $S^{\perp}$ is a subspace of $\mathbb{R}^n$.

7.7 Orthogonal sets and bases

Let V be an inner product space. A subset S $=\{u_1, u_2, \ldots u_n\}$ of nonzero vectors of V is called an orthogonal set if every pair of vectors are orthogonal, i.e.

$$< u_i, u_j > = 0 \ for \ 1 \le i, j \le n$$

The set S $=\{u_1, u_2, \ldots u_n\}$ is said to be orthonormal if $\|u\| = 1$

Theorem 7.1 (Pythagoras). Let $v1, v2, \ldots, vk$ be mutually orthogonal vectors. Then

$$\|v_1 + v_2 + \cdots + v_k\|^2 \le \|v_1\|^2 + \|v_2\|^2 + \cdots + \|v_k\|^2$$

Theorem 7.2. *Let $v1, v2, \ldots, vk$ be an orthogonal basis of a subspace W. Then for any $w \in W$,* $w = \frac{<v_1,w>}{<v_1,v_1>} v_1 + \frac{<v_2,w>}{<v_2,v_2>} v_2 + \ldots + \frac{<v_n,w>}{<v_n,v_n>} v_n$

7.8 Orthogonal projection

Definition: Orthogonal projection is a method of mapping a vector onto a subspace in such a way that the error (difference between the original vector and the projection) is minimized and is orthogonal (perpendicular) to the subspace. It is widely used in linear algebra, geometry, and computer graphics.

Mathematical Concepts: Let V be a vector space with an inner product (dot product in Euclidean space). The **orthogonal projection** of a vector **v** onto a subspace W is the vector in W that is closest to **v.**

Projection onto a Line

Given a nonzero vector **a** that defines a line through the origin, the **orthogonal projection** of a vector **v** onto **a** is:

$$\boldsymbol{Proj_a(v)} = \frac{\boldsymbol{v.a}}{\boldsymbol{a.a}}\boldsymbol{a}$$

Where:

- **v·a** is the dot product of v and **a**.
- **a.a** is the squared magnitude of **a**.
- The result is a vector parallel to **a**.

Projection onto a Subspace

If W is a subspace spanned by an orthonormal set of vectors $\{u_1, u_2, \dots, u_n\}$, the orthogonal projection of v onto W is:

$$Proj_W v = \sum_{i=1}^{n} (v.u_i)u_i$$

This formula ensures that the projection remains within the subspace.

Properties of Orthogonal Projection

Idempotency: Applying the projection twice gives the same result:

$$Proj_W(Proj_W \boldsymbol{v}) = Proj_W \boldsymbol{v}$$

Minimal Distance Property: The difference between v and its projection $Proj_W \boldsymbol{v}$ is the shortest possible.

Orthogonality Condition: The error vector $e = v - Proj_W \boldsymbol{v}$ is orthogonal to every vector in W.

Linearity: Projection is a linear transformation.

Projection Matrix

For a subspace defined by an **orthonormal basis** $U = [u_1, u_2, \dots, u_n]$ the **projection matrix** is: P$P = UU^T$.

For a general subspace with basis vectors in matrix form A, the projection matrix is:

$P = A (A^T A)^{-1} A^T$.

Applications of Orthogonal Projection

- Least Squares Approximation: Used in regression to find the best-fitting line or plane.
- Computer Graphics: Used in rendering and perspective transformations.
- Signal Processing: Used in noise reduction and filtering.
- Machine Learning: Principal Component Analysis (PCA) relies on orthogonal projection to reduce dimensions.
- Structural Engineering: Analyzing force components in different directions.

Example 7.8.1 Given: v = [3, 4], a = [1, 2]

1. Compute the dot products:

v · a = (3)(1) + (4)(2) = 3 + 8 = 11

a · a = (1)(1) + (2)(2) = 1 + 4 = 5

2. Compute the projection:

$Proj_W \boldsymbol{v} = \frac{11}{5} * [1, 2] = [\frac{11}{5}, \frac{22}{5}]$

7.8.1 Gram-Schmidt Orthogonalization Process

The **Gram-Schmidt orthogonalization** process is a method used in linear algebra to convert a set of linearly independent vectors into an orthogonal (or orthonormal) set of vectors while preserving their span. It is widely used in numerical analysis, quantum mechanics, and signal processing.

Mathematical Concept: Given a set of linearly independent vectors $\{v_1, v_2, \ldots, v_n\}$ in an inner product space (such as $\mathbb{R}^n$), the Gram-Schmidt process constructs an orthogonal set $\{u_1, u_2, \ldots, u_n\}$, and if normalized, an orthonormal set $\{e_1, e_2, \ldots, e_n\}$, where:

$$e_i = \frac{u_i}{||ui||}$$

Gram-Schmidt Algorithm

Given a set of linearly independent vectors $\{v_1, v_2, \ldots, v_n\}$

- Initialize the first orthogonal vector: $u_1 = v_1$
- Iterate for each vector v_i and subtract projections:

$$u_i = v_i - \sum_{j=1}^{i-1} Proj_{u_j} v_i \ (for\ j = 1\ to\ i - 1)$$

where $Proj_{u_j} v_i = \frac{v_i . u_j}{u_j . u_j} u_j$

- Normalization (Optional) to obtain an orthonormal set:

$$e_i = \frac{u_i}{||ui||}$$

Example 7.8.2 Given: v1 = [3,1], v2 = [2,2]

Step 1: Compute First Orthogonal Vector

u1 = v1 = [3,1]

Step 2: Compute Projection of v_2 onto u_1

$$Proj_{u_1} v_2 = (\frac{v_2 \cdot u_1}{u1 \cdot u1}) * u1$$

= (6 + 2) / (9 + 1) * [3,1] = [2.4, 0.8]

Step 3: Compute Second Orthogonal Vector

$$u_2 = v_2 - Proj_{u_1} v_2 = [2,2] - [2.4,0.8] = [-0.4, 1.2]$$

Step 4: Normalize to Obtain an Orthonormal Set (Optional)

$$e_1 = \frac{u_1}{||u_1||} = [\frac{3}{\sqrt{(10)}}, \frac{1}{\sqrt{(10)}}]$$

$$e_2 = \frac{u_2}{||u_2||} = [-\frac{0.4}{\sqrt{1.6}}, \frac{1.2}{\sqrt{1.6}}]$$

Properties of Gram-Schmidt Process

- Orthogonality: The resulting set of vectors is orthogonal.
- Preserves Span: The new vectors span the same subspace as the original set.
- Numerical Stability: It can suffer from rounding errors in high dimensions.
- Sequential Computation: Each new vector depends on previous ones, making parallelization difficult.

Applications of Gram-Schmidt Orthogonalization

- QR Decomposition: Used to decompose a matrix A into an orthogonal matrix Q and an upper triangular matrix R.
- Principal Component Analysis (PCA): Used to create orthonormal bases in data analysis.
- Signal Processing: Helps in constructing orthogonal signals.
- Solving Least Squares Problems: Useful in linear regression models.
- Quantum Mechanics: Used in orthogonalization of quantum states

7.9 Orthogonal Matrices

Definition: An **orthogonal matrix** is a square matrix Q

with real entries whose columns and rows are orthogonal unit vectors (i.e., orthonormal vectors). Mathematically, it satisfies the condition:

$$QQ^T = Q^TQ = I$$

Where:

i. Q^T is the transpose Q

ii. I is the Identity Matrix.

Key Properties:

1. **Inverse Equals Transpose:**
   The inverse of an orthogonal matrix Q is its transpose:

   $$Q^{-1} = Q^T$$

   This makes orthogonal matrices easy to invert.
2. **Preservation of Dot Product:**
   For any vectors **x** and **y**, the dot product is preserved under multiplication by QQ:
   $$(Qx) \cdot (Qy) = \boldsymbol{x} \cdot \boldsymbol{y}$$

3. **Preservation of Norm:**
   The Euclidean norm (length) of a vector is preserved:
   $$\| Q\boldsymbol{x} \| = \| \boldsymbol{x} \|$$

   This implies that orthogonal matrices represent linear transformations that are isometries (distance-preserving).
4. **Determinant:**
   The determinant of an orthogonal matrix is either +1+1 or −1−1:
   det(Q)=±1

   1. If det(Q)=1, Q represents a rotation.
   2. If det(Q)=−1, Q represents a reflection or a rotation combined with a reflection.
5. **Eigenvalues:**
   The eigenvalues of an orthogonal matrix lie on the unit circle in the complex plane, meaning they have absolute value 1.
6. **Orthonormal Columns and Rows:**
   The columns and rows of Q form an orthonormal set:
   $$\boldsymbol{q}_i \cdot \boldsymbol{q}_j = \delta_{ij}$$

   where δij is the Kronecker delta.

Applications:

1. **Rotation and Reflection:**
   Orthogonal matrices are used to represent rotations and reflections in geometry and computer graphics.
2. **QR Decomposition:**
   In numerical linear algebra, orthogonal matrices are used in QR decomposition, which is a method for solving linear systems and eigenvalue problems.
3. **Signal Processing:**
   Orthogonal matrices are used in signal processing for transformations like the Discrete Fourier Transform (DFT) and wavelet transforms.
4. **Principal Component Analysis (PCA):**
   In statistics and machine learning, orthogonal matrices are used in PCA to reduce the dimensionality of data while preserving variance.

Example 7.9.1: 2D Rotation Matrix:
A 2D rotation matrix that rotates vectors by an angle θ is:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

This matrix is orthogonal since $Q^T Q = QQ^T = I$.

1. **Identity Matrix:**
   The identity matrix I is trivially orthogonal.
2. **Householder Reflection:**
   A Householder matrix, used in numerical algorithms, is an orthogonal matrix that reflects vectors across a hyperplane.

Theorem 7.9.1. A linear transformation $T : V \to V$ is an isometry if and only if T preserving inner product, i.e., for $\mathbf{u}, \mathbf{v} \in V$,

$$\langle T(\boldsymbol{u}), T(\boldsymbol{v})\rangle = \langle \boldsymbol{u}, \boldsymbol{v}\rangle.$$

Proof. Note that for vectors $\boldsymbol{u}, \boldsymbol{v} \in V$,

$$\|T(\boldsymbol{u}+\boldsymbol{v})\|^2 = \langle T(\boldsymbol{u}+\boldsymbol{v}), T(\boldsymbol{u}+\boldsymbol{v})\rangle = \langle T(\boldsymbol{u}), T(\boldsymbol{u})\rangle + \langle T(\boldsymbol{v}), T(\boldsymbol{v})\rangle + 2\langle T(\boldsymbol{u}), T(\boldsymbol{v})\rangle$$
$$= T(\boldsymbol{u})\|^2 + \|T(\boldsymbol{v})\|^2 + 2\langle T(\boldsymbol{u}), T(\boldsymbol{v})\rangle,$$

$$\|\boldsymbol{u}+\boldsymbol{v}\|^2 = \langle \boldsymbol{u}+\boldsymbol{v}, \boldsymbol{u}+\boldsymbol{v}\rangle = \langle \boldsymbol{u}, \boldsymbol{u}\rangle + \langle \boldsymbol{v}, \boldsymbol{v}\rangle + 2\langle \boldsymbol{u}, \boldsymbol{v}\rangle = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 + 2\langle \boldsymbol{u}, \boldsymbol{v}\rangle.$$

It is clear that the length preserving is equivalent to the inner product preserving.

Theorem 7.9.2. *Let Q be an $n \times n$ matrix. The following are equivalent.*

(a) *Q is orthogonal.*

(b) *Q^T is orthogonal.*

(c) *The column vectors of Q are orthonormal.*

(d) *The row vectors of Q are orthonormal.*

Proof is beyond the book.

Theorem 7.9.2. Let V be an n-dimensional inner product space with an orthonormal basis B = $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$. Let T: $V \to V$ be a linear transformation. Then T is an isometry if and only if the matrix of T relative to B is an orthogonal matrix.

Proof. Let A be the matrix of T relative to the basis B. Then

$$[T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), \ldots, T(\boldsymbol{u}_n)] = [\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n]A.$$

Note that T is an isometry if and only if $\{T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), \ldots, T(\boldsymbol{u}_n)\}$ is an orthonormal basis of V, and that $T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), \ldots, T(\boldsymbol{u}_n)$ is an orthonormal basis if and only the transition matrix A is an orthogonal matrix.

7.10 Diagonalization of real symmetric matrices

Let V be an n-dimensional real inner product space. A linear mapping $T : V \to V$ is said to be **symmetric** if

$$\langle T(\boldsymbol{u}), \boldsymbol{v}\rangle = \langle \boldsymbol{u}, T(\boldsymbol{v})\rangle \; for\ all\ \boldsymbol{u}, \boldsymbol{v} \in V.$$

Example 7.10.1. Let A be a real symmetric $n \times n$ matrix. Let $T : \mathrm{R}^n \to \mathrm{R}^n$ be defined by $T(\boldsymbol{x}) = A\boldsymbol{x}$. Then
T is symmetric for the Euclidean n-space. In fact, for $\boldsymbol{u}, \boldsymbol{v} \in \mathrm{R}^n$, we have

$$T(\boldsymbol{u}) \cdot \boldsymbol{v} = (A\boldsymbol{u}) \cdot \boldsymbol{v} = (A\boldsymbol{u})^T \boldsymbol{v} = \boldsymbol{u}^T A^T \boldsymbol{v}$$
$$= \boldsymbol{u}^T A\boldsymbol{v} = \boldsymbol{u} \cdot A\boldsymbol{v} = \boldsymbol{u} \cdot T(\boldsymbol{v}).$$

Proposition 7.10.1. Let V be an n-dimensional real inner product space with an orthonormal basis $\mathrm{B} = \{\mathrm{u}_1, \mathrm{u}_2 \ldots \mathrm{u}_\mathrm{n}\}$. Let $\mathrm{T} : \mathrm{V} \to \mathrm{V}$ be a linear mapping whose matrix relative to B is A. Then T is symmetric if and only the matrix A is symmetric.

Proof. Note that

$$[T(u_1), T(u_2) \ldots . T(u_n)] = [u_1, u_2, \ldots u_n] \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

Alternatively,

$$T(u_j) = \sum_{i=1}^{n} a_{ij} u_i \,, for\ 1 \le j \le n$$

If T is symmetric, then

$$a_{ij} =< u_i, T(u_j) >=< T(u_i), u_j > = a_{ji}$$

So, A is symmetric.

Conversely, if A is symmetric, then for vectors $u = \sum_{i=1}^{n} a_i u_i$, $v = \sum_{i=1}^{n} b_i u_i$ we have

$$< T(u), v > = \sum_{i,j=1}^{n} a_i b_j T(u_i) u_j = \sum_{i,j=1}^{n} a_i b_j a_{ji}$$

$$= \sum_{i,j=1}^{n} a_i b_j a_{ij} =< u, T(v) >$$

Therefore, T is symmetric.

7.11 Illustrated examples

1. Find the projection of u = (3, 4) onto v = (1, 2).

➢ The projection formula is: $Proj_W\boldsymbol{v} = (\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{||\boldsymbol{v}||^2}\boldsymbol{v}$

Compute the dot product: u · v = (3 × 1) + (4 × 2) = 3 + 8 = 11

Find $||v||^2 = (1^2 + 2^2) = 1 + 4 = 5$

$||\boldsymbol{v}||^2$ = (11/5) . (1, 2) = (11/5, 22/5)

2. Verify whether the set of vectors (1, 0, -1), (0, 1, 1), and (1, 1, 0) forms an orthogonal set.

➢ Vectors are orthogonal if every pair has a dot product of zero.

Check $v_1 \cdot v_2 = (1 \times 0) + (0 \times 1) + (-1 \times 1) = 0 + 0 - 1 = -1 \neq 0$

Since $v_1 \cdot v_2 \neq 0$, the set is not orthogonal.

3. Use the Gram-Schmidt process to orthogonalize the set {(1,1), (1,-1)}.

Solution:

Let $v_1 = (1,1)\ and\ v_2 = (1,-1)$.

$$Set\ u_1 = v_1 = (1,1).$$

Compute projection of v2 onto u1: $Proj_{u_1}\boldsymbol{v_2} = \left(\frac{v_2 \cdot u_1}{u_1 \cdot u_1}\right)u_1$

= ((1 × 1) + (-1 × 1)) / ((1 × 1) + (1 × 1)) (1,1) = (0,0)

Compute $u_2 = v_2 - Proj_{u1}(v2) = (1,-1) - (0,0) = (1,-1)$

Thus, the orthogonal set is {(1,1), (1,-1)}, which is already orthogonal.

4. Find a unit vector orthogonal to both a = (1,2,3) and b = (4,5,6).

➢ A vector orthogonal to both can be found using the cross product:

$$a \times b = \begin{bmatrix} i & j & k \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$= i(2 \times 6 - 3 \times 5) - j(1 \times 6 - 3 \times 4) + k(1 \times 5 - 2 \times 4)$$

$$= (-3, 6, -3)$$

Now, find the unit vector: $||a \times b|| = \sqrt{((-3)^2 + 6^2 + (-3)^2)} = \sqrt{(54\,)} = 3\sqrt{6}$

Unit vector = $(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6})}$).

5. Compute the inner product of u = (1, 2, -1) and v = (3, 0, 4) in R^3.

   - The inner product (dot product) is given by:

$$\langle u, v\rangle = (1 \times 3) + (2 \times 0) + (-1 \times 4)$$

$$= 3 + 0 - 4 = -1$$

Thus, $\langle u, v\rangle = -1$.

6. Show that the function $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$ dx defines an inner product.

   - To be an inner product, $\langle f, g\rangle$ must satisfy:

i. Linearity: $\langle af + bg, h\rangle = a\langle f, h\rangle + b\langle g, h\rangle$ (holds by properties of integrals).
ii. Symmetry: ⟨f, g⟩ = ⟨g, f⟩ (holds because multiplication is commutative).
iii. Positivity: ⟨f, f⟩ ≥ 0 and ⟨f, f⟩ = 0 if and only if f(x) = 0 (follows from integral properties).

Since all conditions hold, this defines a valid inner product.

7. Find the norm of the vector u = (3, -4, 12) in the inner product space R^3.

   - The norm is given by $||u|| = \sqrt{(\langle u, u\rangle)}$.

$$\langle u, u\rangle = (3^2 + (-4)^2 + 12^2) = 9 + 16 + 144 = 169.$$

$$||u|| = \sqrt{(169)} = 13.$$

Thus, the norm of u is 13.

8. Find the angle between the vectors u = (1, 2, 2) and v = (2, 1, 3) in an inner product space.

   - The angle θ is given by $cos(\theta) = \frac{\langle u, v\rangle}{(||u||\,||v||)}$.

⟨u, v⟩ = (1×2) + (2×1) + (2×3) = 2 + 2 + 6 = 10.

$$||u|| = \sqrt{(1^2 + 2^2 + 2^2)} = \sqrt{9} = 3.$$

$$||v|| = \sqrt{(2^2 + 1^2 + 3^2)} = \sqrt{14}.$$

$$cos(\theta) = \frac{10}{3\sqrt{14}}$$

Thus, the angle can be found using

$$\theta = \cos^{-1}\frac{10}{3\sqrt{14}}.$$

9. Verify if the set {(1,1), (-1,1)} is an orthonormal set in R^2 under the standard inner product.

   - A set is orthonormal if each vector has norm 1 and they are mutually orthogonal.

Find norms: $||(1,1)|| = \sqrt{(1^2 + 1^2)} = \sqrt{(2)}, ||(-1,1)|| = \sqrt{((-1)^2 + 1^2)} = \sqrt{(2)}$.

Since norms are not 1, normalize: $u_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), u_2 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

Check orthogonality: $\langle u_1, u_2\rangle = (1/\sqrt{2})(-1/\sqrt{2}) + (1/\sqrt{2})(1/\sqrt{2}) = -1/2 + 1/2 = 0$.

Since they are unit vectors and orthogonal, they form an orthonormal set.

Exercises

- <u>MCQ type questions:</u>

1. Two vectors and in an inner product space are orthogonal if:

   A) $u.v = 0$

   B) $u.v = 1$

   C) $u.v = -1$

   D) $u.v > 0$

2. In Euclidean space $\mathbb{R}^n$, the dot product of two vectors and is given by:

   A) $\sum a_i b_i$

   B) $\sum a_i^2 + \sum b_i^2$

   C) $\sum a_i b_i^2$

   D) $\sum (a_i + b_i)$

3. If a set of nonzero vectors is mutually orthogonal, it is called:

   A) An independent set

   B) A unit vector set

   C) An orthogonal set

   D) A basis

4. An orthonormal set is an orthogonal set where each vector is:
   A) A unit vector
   B) Linearly dependent
   C) Equal to zero
   D) Parallel to others
5. The Gram-Schmidt process is used to:
   A) Compute determinants
   B) Convert a set of vectors into an orthonormal basis
   C) Solve linear equations
   D) Compute eigenvalues
6. The projection of a vector onto another vector is given by:
   A) $\frac{u.v}{||v||}$
   B) $\frac{u.v}{||v||}$v
   C) $||u||\,||v||$
   D) $u + v$
7. In an inner product space, the norm of a vector is given by:
   A) $||v|| = v.v$
   B) $||v|| = \sqrt{v.v}$
   C) $||v|| = \sum v_i$
   D) $||v|| = v.v^2$
8. If two vectors and are orthogonal, their angle satisfies:
   A) $\theta = 0°$
   B) $\theta = 90°$
   C) $\theta = 180°$
   D) $\theta = 45°$
9. The standard basis vectors in $\mathbb{R}^3$, $e_1 = (1,0,0),\ e_2 = (0,1,0),\ e_3 = (0,0,1)$ are:
   A) Orthogonal but not orthonormal
   B) Orthonormal
   C) Linearly dependent
   D) Zero vectors
10. If and are orthogonal, their dot product is:
   A) Always positive

B) Always negative

C) Always zero

D) Undefined

- Short Answer type Questions:

1. Define an inner product space with an example.
2. What are the properties of an inner product?
3. State the Cauchy-Schwarz inequality.
4. What is the norm of a vector in an inner product space?
5. When are two vectors said to be orthogonal?
6. State and explain the Pythagorean theorem in inner product spaces.
7. Define an orthonormal set. Give an example.
8. What is the Gram-Schmidt process used for?
9. If ⟨u, v⟩ = 0, what does that imply about the vectors u and v?
10. What is the geometric significance of orthogonality in $\mathbb{R}^2$ or $\mathbb{R}^3$?
11. Find the projection of u = (9, 11) onto v = (1, 0).
12. Given vectors u=(1,2), v=(3,4), compute the inner product ⟨u,v⟩ in $\mathbb{R}^2$ using the standard inner product.
13. Find the norm of the vector v=(3,4,0) in $\mathbb{R}^3$ with the standard inner product.
14. Determine whether the vectors u=(1,2,3) and v=(2,−1,0) are orthogonal in $\mathbb{R}^3$.
15. Verify the Pythagorean theorem for the vectors u=(2,1), v=(−1,2) in $\mathbb{R}^2$.

Answers:

MCQ type questions:

1-A, 2-A, 3-C, 4-A, 5-B, 6-B, 7-B, 8-B, 9-B, 10-C.

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